Projective Resolutions and Yoneda Algebras for Algebras of Dihedral Type.

Alexander Generalov
Chair of Higher Algebra, Dept. of Mathematics
Saint-Petersburg State University, Universitetskiy pr. 28
198504 Saint-Petersburg Russia
general@pdmi.ras.ru

Nikolai Kosmatov
Faculté des Sciences, Université de Franche-Comté
16 route de Gray, 25030 Besançon France
kosmatov@lifc.univ-fcomte.fr

January 25, 2006

Abstract

This paper provides a method for the computation of Yoneda algebras for algebras of dihedral type. The Yoneda algebras for one infinite family of algebras of dihedral type (the family $D(3\mathcal{R})$ in K. Erdmann's notation) are computed. The minimal projective resolutions of simple modules were calculated by an original computer program implemented by one of the authors in C++ language. The algorithm of the program is based on a diagrammatic method presented in this paper and inspired by that of D. Benson and J. Carlson.

Keywords: Yoneda algebra, algebras of dihedral type, projective resolutions, module diagrams.

1 Introduction

The algebras of dihedral, semidihedral and quaternion type were defined and classified by Karin Erdmann in [1]. They generalize the blocks with dihedral, semidihedral and generalized quaternion defect groups respectively.

The classification contains dozens of infinite families of algebras. Each family is defined by a quiver with relations containing some parameters.

The Yoneda algebras of some dihedral and semidihedral algebras were computed by the first author et al. in [2]–[8]. This computation contains two steps: to find the projective resolutions of simple modules, and to determine the Yoneda algebra. For the algebras that appear as principal blocks of group algebras, these results allowed to find the cohomology ring of the corresponding groups.

It turns out that for all considered algebras, the minimal projective resolution of a simple module is periodic or can be represented as the total complex of an infinite bicomplex. The bicomplex repeats itself in some regular way, but in general is not periodic. To find the structure of the bicomplex, it is often necessary to determine its first 10–20 diagonals. This computation being rather difficult to do by hand, the object of this work is not only to find the Yoneda algebras for other families of dihedral algebras, but also to use computer-based techniques to find the projective resolutions.

In this paper, we give a general description of our method for the computation of Yoneda algebras for the dihedral algebras. We apply this method to determine the Yoneda algebras for one infinite family of dihedral algebras: the family $D(3\mathcal{R})$ in the notation of [1]. The projective resolutions for this family were computed by an original C++ program *Resolut* [9] implemented by the second author. The computations made for other dihedral algebras show that this program can be also efficiently used for most of them.

The algorithm of the program is based on a diagrammatic method inspired by that of David Benson and Jon Carlson [10]. Although our definition of a diagram is different from those of [10, 11, 12], many ideas and diagram constructions of [10] still apply in our case. An important advantage of our approach is the possibility to implement a significant part of the Yoneda algebra computation in a computer program. We define a diagram of a module with respect to a basis of this module. This definition gives a simple tool to examine the structure of projective modules, to compute the projective resolutions and to prove our results. We describe in terms of diagrams all modules, homomorphisms, kernels and images that appear in our computation. It allows to consider diagrams instead of modules and diagram maps instead of homomorphisms.

The program examines the algebra defined by the given quiver with relations (with fixed values of parameters) and computes the minimal projective resolutions of the simple modules over this algebra. For every simple module S_i, the program tries to construct a bicomplex lying in the first quadrant of the plane and consisting of projective modules, such that its total complex gives the minimal projective resolution of S_i. After computing a new diagonal of the bicomplex, the program compares the dimensions of the corresponding image and kernel in the total complex to check the exactness.

It takes less than one second to compute sufficiently many modules in the bicomplex to see its structure. Running the program for different parameters allows to conjecture the general form of the bicomplex for arbitrary parameters. The conjecture is easy to prove by hand, as the bicomplex contains only finitely many different squares. A more complete presentation of the program will be the object of a separate article.

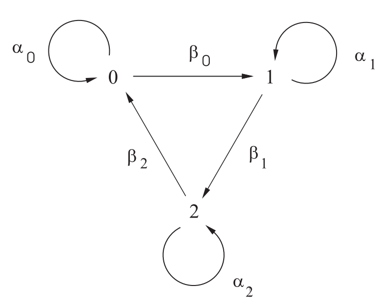
The paper is organized as follows. In Section 2, we define the family $D(3\mathcal{R})$ of dihedral algebras, state our main result and describe our method of computation of Yoneda algebras. Section 3 introduces the notion of a diagram and provides some properties of diagrams. In Section 4, we apply the diagrammatic method to compute the minimal projective resolutions and syzygies of simple modules. We define the generators of the Yoneda algebra in Section 5 and complete the proof of our main result in Section 6.

2 Main Result

Let K be a field, Λ be an associative K-algebra with identity, M be a Λ-module (all the considered modules are left modules). The K-module $\mathcal{E}xt(M) = \bigoplus_{m\geqslant 0} \mathrm{Ext}^m_\Lambda(M,M)$ can be endowed with the structure of an associative K-algebra using the Yoneda product [13]. The algebra $\mathcal{E}xt(M)$ is called *the* Ext-*algebra of* M.

If Λ is a basic finite dimensional K-algebra, we set $\overline{\Lambda} = \Lambda/J(\Lambda)$ where $J(\Lambda)$ is the Jacobson radical of Λ. The Ext-algebra $\mathcal{E}xt(\overline{\Lambda})$ is called *the Yoneda algebra of* Λ and is denoted by $\mathcal{Y}(\Lambda)$.

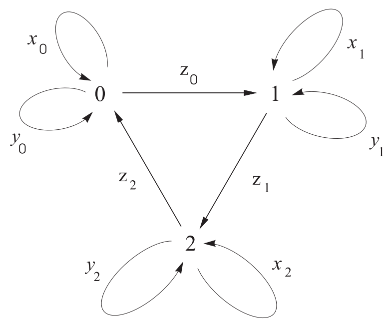
Let k, s, t, u be integers such that $k \geqslant 1$ and $s, t, u \geqslant 2$. We define the K-algebra $R_{k,s,t,u}$ by the following quiver with relations (we write down a composition from the right to the left):

Q: 

,

$$\begin{aligned} &\beta_0\alpha_0 = \beta_1\alpha_1 = \beta_2\alpha_2 = 0, \quad \alpha_1\beta_0 = \alpha_2\beta_1 = \alpha_0\beta_2 = 0, \\ &\alpha_0^s = (\beta_2\beta_1\beta_0)^k, \quad \alpha_1^t = (\beta_0\beta_2\beta_1)^k, \quad \alpha_2^u = (\beta_1\beta_0\beta_2)^k. \end{aligned} \tag{2.1}$$

The algebras $R_{k,s,t,u}$ compose an infinite family of dihedral algebras, which is denoted in [1] by $D(3\mathcal{R})$. Every $R_{k,s,t,u}$ is a symmetric algebra (and therefore a QF-algebra). To describe the Yoneda algebra $\mathcal{Y}(R_{k,s,t,u})$, let us consider the quiver T:

T:  .

Let $K[T]$ be the path algebra of T. We define the following grading on $K[T]$:

$$\deg(x_i) = \deg(z_i) = 1,\ \deg(y_i) = 2,\ i = 0, 1, 2.$$

Consider the following relations on the quiver T:

$$\begin{gathered} z_1z_0 = z_2z_1 = z_0z_2 = 0, \\ x_0y_0 = y_0x_0,\ x_1y_1 = y_1x_1,\ x_2y_2 = y_2x_2, \\ z_0y_0 = y_1z_0,\ z_1y_1 = y_2z_1,\ z_2y_2 = y_0z_2, \\ x_0^2 = \delta(s, 2)y_0,\ x_1^2 = \delta(t, 2)y_1,\ x_2^2 = \delta(u, 2)y_2. \end{gathered} \tag{2.2}$$

Here $\delta(i, j)$ denotes the Kroneker delta function: $\delta(i, j) = 1$ if $i = j$, and 0 otherwise.

Let $\mathcal{E}_{k,s,t,u}$ be the K-algebra defined by the quiver T with the relations (2.2). As all these relations are homogeneous, the algebra $\mathcal{E}_{k,s,t,u}$ inherits a grading from $K[T]$. We can now state our main result.

Theorem 2.1. *The Yoneda algebra $\mathcal{Y}(R_{k,s,t,u})$ is isomorphic, as a graded algebra, to $\mathcal{E}_{k,s,t,u}$.*

To simplify notation, we set $R = R_{k,s,t,u}$, $\mathcal{E} = \mathcal{E}_{k,s,t,u}$ and $\mathcal{Y} = \mathcal{Y}(R)$. We denote by e_i the idempotents of R corresponding to the vertices $i = 0, 1, 2$ of Q. There exist three indecomposable projective R-modules and three simple R-modules (up to isomorphism), they are defined by $P_i = Re_i$ and $S_i = P_i/(J(R)P_i)$, respectively.

Let us now describe our method of computation of the Yoneda algebras. This method uses the technique of D.Benson and J.Carlson [10] with some essential improvements. It can be also applied to other families of dihedral algebras defined in K. Erdmann's classification [1].

1) We examine the given quiver with relations Q to find the bases and the diagrams of the indecomposable projective modules.
2) Using the diagrammatic method, we compute the bicomplexes such that their total complexes give minimal projective resolutions of simple modules. We also describe the syzygies in terms of diagrams. The first two steps can be done today by the program *Resolut* [9].
3) We chose some generators in the groups $\mathrm{Ext}^1_R(S_i, S_j)$ and check if they generate the groups $\mathrm{Ext}^2_R(S_i, S_j)$ in the Yoneda algebra. If not, we chose additional generators in $\mathrm{Ext}^2_R(S_i, S_j)$ and so on, until the generators seem to generate the Yoneda algebra.
4) Computing the products of the generators, we find the relations and conjecture a quiver with relations defining the Yoneda algebra. The conjecture is proved as it is shown below.

3 Diagrams

Set $L = \{\alpha_k, \beta_k \,|\, k = 0, 1, 2\}$. (More generally, if R is the algebra defined by a quiver with relations Q, let L be the set of edges of Q.) Let M be an R-module and $D = (V, E, \lambda)$ be a finite directed graph with vertices V, edges E and a labelling function $\lambda : E \to L$. If $i, j \in V$ and $e \in E$ is an edge $i \to j$ with the label $\lambda(e) = \gamma \in L$, we write $e = e(i, j)$ or $e = e(i, j, \gamma)$.

Definition 3.1. We say that M *has a diagram* D if there exists a K-basis $\{v_i \,|\, i \in V\}$ of M such that
(i) for any edge $e(i, j, \gamma)$, we have $\gamma v_i = v_j$ or $\gamma v_i = -v_j$,
(ii) for any $i \in V$ and $\gamma \in L$ with $\gamma v_i \neq 0$, there exists a unique $j \in V$ such that $e(i, j, \gamma) \in E$,
(iii) for any v_i, the R-module $\mathrm{top}\,(Rv_i)$ is simple, i.e. Rv_i is a local module.

The same module M can have different diagrams according to this definition. As we consider the diagrams with respect to some fixed bases, we do not need the diagram uniqueness in our results. For simplicity of notation, we assume that a non-directed edge in a diagram denotes an arrow from the higher vertex to the lower one, and we write sometimes just $i \in D$ instead of $i \in V$. It is convenient to write the simple module $\mathrm{top}\,(Rv_i)$ in the vertex i of the diagram.

To give an example of diagrams, let us determine the diagrams of P_i. It is easily seen from (2.1) that $P_0 = Re_0$ has the K-basis

$$\begin{gathered} e_0,\ \alpha_0,\ \alpha_0^2, \dots,\ \alpha_0^{s-1}, \\ \beta_0,\ \beta_1\beta_0,\ \beta_2\beta_1\beta_0,\ \beta_0\beta_2\beta_1\beta_0, \dots,\ \beta_1\beta_0(\beta_2\beta_1\beta_0)^{k-1}, \\ \alpha_0^s = (\beta_2\beta_1\beta_0)^k. \end{gathered} \tag{3.1}$$

The modules P_1 and P_2 have analogous bases which can be obtained from (3.1) by a circular permutation of indices and a simultaneous permutation of parameters:

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ & \nwarrow \quad \swarrow & \\ & 2 & \end{array}, \qquad \begin{array}{ccc} s & \longrightarrow & t \\ & \nwarrow \quad \swarrow & \\ & u & \end{array}. \tag{3.2}$$

Due to the obvious symmetry of the quiver Q and the relations (2.1) with respect to such permutations, it is sufficient to state and to prove the majority of our results for P_0 and S_0 only.

We obtain using (3.1) that the modules P_0, P_1 and P_2 have the diagrams

$$\begin{array}{ccc} & S_0 & \\ {}^{\alpha_0}\swarrow & & \searrow^{\beta_0} \\ S_0 & & S_1 \\ |\alpha_0 & & |\beta_1 \\ S_0 & & S_2 \\ |\alpha_0 & & |\beta_2 \\ S_0 & & S_0 \\ |\alpha_0 & & |\beta_0 \\ \vdots & & \vdots \\ |\alpha_0 & & |\beta_1 \\ S_0 & & S_2 \\ {}_{\alpha_0}\searrow & & \swarrow_{\beta_2} \\ & S_0 & \end{array}, \quad \begin{array}{ccc} & S_1 & \\ {}^{\alpha_1}\swarrow & & \searrow^{\beta_1} \\ S_1 & & S_2 \\ |\alpha_1 & & |\beta_2 \\ S_1 & & S_0 \\ |\alpha_1 & & |\beta_0 \\ S_1 & & S_1 \\ |\alpha_1 & & |\beta_1 \\ \vdots & & \vdots \\ |\alpha_1 & & |\beta_2 \\ S_1 & & S_0 \\ {}_{\alpha_1}\searrow & & \swarrow_{\beta_0} \\ & S_1 & \end{array}, \quad \begin{array}{ccc} & S_2 & \\ {}^{\alpha_2}\swarrow & & \searrow^{\beta_2} \\ S_2 & & S_0 \\ |\alpha_2 & & |\beta_0 \\ S_2 & & S_1 \\ |\alpha_2 & & |\beta_1 \\ S_2 & & S_2 \\ |\alpha_2 & & |\beta_2 \\ \vdots & & \vdots \\ |\alpha_2 & & |\beta_0 \\ S_2 & & S_1 \\ {}_{\alpha_2}\searrow & & \swarrow_{\beta_1} \\ & S_2 & \end{array} \tag{3.3}$$

respectively. Set

$$\begin{gathered} b_0 = (\beta_2\beta_1\beta_0)^{k-1},\ b_1 = (\beta_0\beta_2\beta_1)^{k-1},\ b_2 = (\beta_1\beta_0\beta_2)^{k-1}, \\ c_0 = \beta_1\beta_0 b_0,\ c_1 = \beta_2\beta_1 b_1,\ c_2 = \beta_0\beta_2 b_2, \\ g_0 = \alpha_0^{s-1},\ g_1 = \alpha_1^{t-1},\ g_2 = \alpha_2^{u-1}. \end{gathered}$$

We will use the same letters for the elements of the path algebra $K[Q]$ and for their images in R. For abbreviation, we denote a sequence of edges in a diagram by one edge and write the composition of the original edges nearby. The diagrams (3.3) can be written in this notation, for example, as

$$\begin{array}{ccc} & S_0 & \\ {}^{\alpha_0}\swarrow & & \searrow^{\beta_0} \\ S_0 & & S_1 \\ {}_{g_0}\searrow & & \swarrow_{c_1} \\ & S_0 & \end{array}, \quad \begin{array}{ccc} & S_1 & \\ {}^{g_1}\swarrow & & \searrow^{c_1} \\ S_1 & & S_0 \\ {}_{\alpha_1}\searrow & & \swarrow_{\beta_0} \\ & S_1 & \end{array}, \quad \begin{array}{ccc} & S_2 & \\ {}^{\alpha_2}\swarrow & & \searrow^{c_2} \\ S_2 & & S_1 \\ {}_{g_2}\searrow & & \swarrow_{\beta_1} \\ & S_2 & \end{array}.$$

Although our definition of a diagram is different from that of [10], many definitions and diagrammatic constructions from [10] are applicable in our context. We briefly discuss some definitions and properties which will be useful below. Let $D = (V, E, \lambda)$ be a diagram of M and let $\{v_i \,|\, i \in V\}$ be the corresponding basis of M. If another R-module M' has the same diagram D, then $M \simeq M'$. Indeed, if $\{v'_i \,|\, i \in V\}$ is the basis of M' from Definition 3.1, the map $v_i \mapsto v'_i$ gives the R-isomorphism.

We say that $D' = (V', E', \lambda')$ is *a subdiagram of* D if $V' \subset V$, $E' = \{e(i,j) \in E \,|\, i,j \in V'\}$ and $\lambda' = \lambda|_{E'}$. Note that a subdiagram D' of D contains all edges that connect two vertices of D', therefore D' is entirely determined by its set of vertices V'. The K-subspace $\sum_{j \in V'} Kv_j$ generated by $\{v_j \,|\, j \in V'\}$ is not necessarily an R-submodule. The R-submodule generated by $\{v_j \,|\, j \in V'\}$ has a diagram which is the subdiagram of D containing all vertices (and therefore all edges) lying on the paths with origine in V'.

We say that a subdiagram D' of D is *open* if for any vertex $j \in D'$, D' contains all vertices lying on the paths in D with origine j. In this case, the R-submodule $M' \subset M$ generated by $\{v_j \,|\, j \in D'\}$ is equal to $\sum_{j \in D'} Kv_j$ and has the diagram D'. Indeed, $\{v_j \,|\, j \in D'\}$ is the required basis of M'.

Dually, we say that a subdiagram D' of D is *closed* if for any vertex $j \in D'$, D' contains all vertices lying on the paths in D with end j. Let M_0 denote the submodule M_0 generated by $\{v_j \,|\, j \notin D'\}$. The quotient $\overline{M} = M/M_0$ has the diagram D'. Indeed, if $\pi : M \to \overline{M}$ is the canonical projection, the required basis of $\overline{M}$ is given by $\{\pi(v_j) \,|\, j \in D'\}$. The element $\pi(v_j) \in \overline{M}$ will be also denoted by $\overline{v_j}$.

As a subdiagram is determined by its set of vertices, we can define the set theoretic operations for the subdiagrams of D by the corresponding operations on their sets of vertices. The open subdiagrams define a topology on the (finite) set of subdiagrams of D, and the open and closed subdiagrams are complementary.

Suppose $V = V_1 \bigsqcup V_2$ and define $M_1 = \sum_{j \in V_1} Kv_j$ and $M_2 = \sum_{j \in V_2} Kv_j$. Then M_1 and M_2 are R-submodules of M iff V_1 and V_2 are not connected (i.e. D has no edge $e(j_1, j_2)$ and no edge $e(j_2, j_1)$ with $j_1 \in V_1, j_2 \in V_2$). Moreover, in this case $M = M_1 \bigoplus M_2$.

The modules $\operatorname{Rad} M$ and $\operatorname{top} M = M/(\operatorname{Rad}(M))$ can be also easily described in terms of subdiagrams. Let D_{Rad} be the open subdiagram of D with vertices $V_{\mathrm{Rad}} = \{j \in V \,|$ there exists an edge $e(i,j)$ in $D\}$, and let D_{top} be the closed subdiagram of D with vertices $V_{\mathrm{top}} = V \backslash V_{\mathrm{Rad}}$ and no edges.

Proposition 3.2. (i) $\operatorname{Rad} M$ *has the diagram* D_{Rad} *with respect to the basis* $\{v_j \,|\, j \in V_{\mathrm{Rad}}\}$. (ii) $\operatorname{top} M$ *has the diagram* D_{top} *with respect to the basis* $\{\overline{v_j} \,|\, j \in V_{\mathrm{top}}\}$.

Proof. We claim that D contains no loops. Otherwise, we have $v_j = \pm\pi v_j$ for some $j \in V$, $\pi = \gamma_1\gamma_2\ldots\gamma_n$, $n \geqslant 1$, $\gamma_i \in L$. As $\pi \in \operatorname{Rad} R$, we have $\pi^l = 0$ for some $l \geqslant 1$, hence $v_j = \pm\pi^l v_j = 0$, a contradiction.

(i) Since D_{Rad} is an open subdiagram of D, it is sufficient to show that $\operatorname{Rad} M = \sum_{j\in V_{\mathrm{Rad}}} Kv_j$. We show first that $\operatorname{Rad} M \subset \sum_{j\in V_{\mathrm{Rad}}} Kv_j$. For any $j \in V_{\mathrm{top}}$, the subdiagram with vertices $V\backslash\{j\}$ is open and defines a submodule $N_j \subset M$. Since $M/N_j \simeq \operatorname{top}(Rv_j)$ is simple, N_j is a maximal submodule of M. It follows that $\operatorname{Rad} M \subset \bigcap_{j\in V_{\mathrm{top}}} N_j = \sum_{j\in V_{\mathrm{Rad}}} Kv_j$.

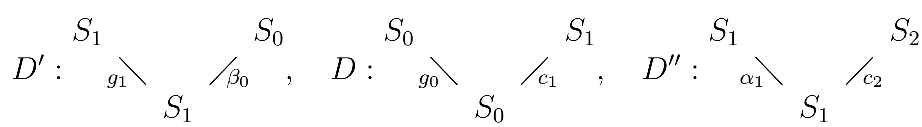
Assume now that $j_1 \in V_{\mathrm{Rad}}$, we claim that $v_{j_1} \in N$ for any maximal submodule $N \subset M$. On the contrary, suppose that $v_{j_1} \notin N$ for a maximal submodule $N \subset M$. Since $j_1 \in V_{\mathrm{Rad}}$, there exists an edge $e(j_0, j_1, \gamma_1)$ in D. Since $v_{j_1} \notin N$ and $v_{j_1} = \pm\gamma_1 v_{j_0}$, we have $v_{j_0} \notin N$. As there are no loops in D, we can find a maximal path $j_0, j_1, j_2, \ldots, j_n$ such that for any $1 \leqslant p \leqslant n$ there is an edge $e(j_{p-1}, j_p, \gamma_p)$ in D and $v_{j_p} \notin N$. This path is maximal iff $\gamma v_{j_n} \in N$ for any $\gamma \in L$. Since N is a maximal submodule of M and $v_{j_n} \notin N$, we have $M = N + Rv_{j_n}$, hence $v_{j_0} \in N + Rv_{j_n}$. It follows that

$$v_{j_n} = \pm\gamma_n \ldots \gamma_1 v_{j_0} \in \gamma_n \ldots \gamma_1(N + Rv_{j_n}) \subset N + \gamma_n \ldots \gamma_1 Rv_{j_n} \subset N,$$

a contradiction.

(ii) The statement for $\operatorname{top} M$ follows from (i) and the definitions. □

Let us consider adjoining of diagrams in a simple case which will be useful below. Let $M' \subset P_1$, $M \subset P_0$ and $M'' \subset P_1$ be the R-submodules defined by the open subdiagrams

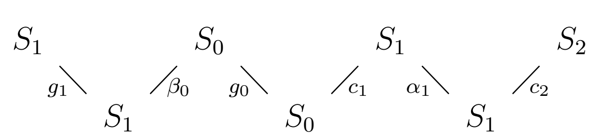


with respect to the bases $\{\alpha_1, \ldots, \alpha_1^t, c_1\} \subset M'$, $\{\alpha_0, \alpha_0^2, \ldots, \beta_1\beta_0, \beta_0\} \subset M$, $\{g_1, c_2\beta_1, \ldots, \beta_1\} \subset M''$ (the basis elements being written from left to right in the same order as the vertices of the diagrams). Then the submodule

$$\begin{array}{c}\langle(\alpha_1,0,0),\ldots,(\alpha_1^t,0,0),(c_1,\alpha_0,0),(0,\alpha_0^2,0),\ldots,\\ (0,\beta_1\beta_0,0),(0,-\beta_0,g_1,),(0,0,c_2\beta_1)\ldots,(0,0,\beta_1)\rangle\end{array} \tag{3.4}$$

of $M' \bigoplus M \bigoplus M'' \subset P_1 \bigoplus P_0 \bigoplus P_1$ has the diagram

$D' + D + D'' :=$



. (3.5)

4 Projective Resolutions

For a simple R-module S_i, let $\ldots \xrightarrow{d_1^{(i)}} Q_1^{(i)} \xrightarrow{d_0^{(i)}} Q_0^{(i)} \xrightarrow{d_{-1}^{(i)}} S_i \to 0$ denote the minimal projective resolution of S_i. We will write $\Omega^n(S_i)$ for its n-th syzygy $\operatorname{Im}(d_{n-1}(M))$, $n \geqslant 0$. The multiplication on the right by an element $x \in e_iRe_j$ induces a homomorphism from P_i into P_j, we denote this homomorphism by the same letter x.

In this section, we show how to use the diagrammatic method to find the minimal projective resolutions and syzygies of S_i. Since the vertices of a diagram of an R-module M correspond to a basis of M and the edges reflect the R-module structure on M, we can consider diagrams and diagram maps rather than modules and homomorphisms. Diagram homomorphisms (obvious in our context) can be formally defined as in [10, Def. 2.6]. Assume $i = 0$. The analogous results for S_1 and S_2 are obtained by permutations (3.2).

We identify S_0 with $\langle \alpha_0^s \rangle \subset P_0$. Set $Q_0^{(0)} = P_0$ and define an epimorphism $d_{-1}^{(0)} : Q_0^{(0)} \to S_0$ by $d_{-1}^{(0)}(e_0) = \alpha_0^s$. Then we have an exact sequence

$$\begin{array}{ccccc} S_0 & & & & S_1 \\ & {}_{g_0}\diagdown & & \diagup_{c_1} & \\ & & S_0 & & \end{array} \hookrightarrow Q_0^{(0)} \xrightarrow{\alpha_0^s} S_0\,,$$

where the open subdiagram on the left represents $\Omega^1(S_0) = \ker d_{-1}^{(0)} \subset P_0$. Set $Q_1^{(0)} = P_0 \bigoplus P_1$ and define an epimorphism $d_0^{(0)} : Q_1^{(0)} \to \Omega^1(S_0)$ by $d_0^{(0)}(e_0, 0) = \alpha_0$ and $d_0^{(0)}(0, e_1) = \beta_0$. We have an exact sequence

$$\begin{array}{ccccccccc} S_1 & & & & S_0 & & & & S_1 \\ & {}_{c_1}\diagdown & & \diagup_{\alpha_0} & & {}_{\beta_0}\diagdown & & \diagup_{g_1} & \\ & & S_0 & & & & S_1 & & \end{array} \hookrightarrow Q_1^{(0)} \xrightarrow{(\alpha_0,\beta_0)} \Omega^1(S_0)\,,$$

where the left diagram represents

$$\begin{aligned} \Omega^2(S_0) = \ker d_0^{(0)} = \langle &(\beta_0, 0), (\beta_1\beta_0, 0), \ldots, ((\beta_2\beta_1\beta_0)^k, 0), \\ &(-g_0, c_1), (0, \beta_0 c_1), (0, \alpha_1^{t-1}), \ldots, (0, \alpha_1)\rangle \subset P_0 \bigoplus P_1\,. \end{aligned}$$

Here $(g_0, -c_1)$ joins two subdiagrams of P_0 and P_1 since $d_0^{(0)}(g_0, 0) = d_0^{(0)}(0, c_1)$. Setting $Q_2^{(0)} = P_1 \bigoplus P_0 \bigoplus P_1$ and $d_1^{(0)} = \begin{pmatrix} \beta_0 & -g_0 & 0 \\ 0 & c_1 & \alpha_1 \end{pmatrix}$, we see that $\Omega^3(S_0) = \ker d_1^{(0)}$ is the submodule (3.4) with the diagram (3.5). Continuing in the same manner and using the induction, the reader will prove the following two propositions.

Proposition 4.1. a) *The diagrams of* $\Omega^0(S_0)$ *and* $\Omega^1(S_0)$ *are, respectively,*

$$S_0 \qquad \text{and} \qquad \begin{array}{ccc} S_0 & & S_1 \\ {}_{g_0}\diagdown & & \diagup_{c_1} \\ & S_0 & \end{array} .$$

b) *Let* $m \geqslant 2$ *be an integer. Suppose* $m \equiv r \pmod 6$ *with* $0 \leqslant r \leqslant 5$. *Let* D *be the diagram of* $\Omega^{m-2}(S_0)$. *Then the diagram of the module* $\Omega^m(S_0)$ *can be obtained from* D *by adjoining some subdiagrams (depending on* r*) on both sides of* D. *The following table shows the subdiagrams to adjoin on the left and on the right side of* D*:*

$r=0$	$\begin{array}{ccc} S_0 & & S_2 \\ {}_{c_0}\diagdown & & \diagup_{\alpha_2} \\ & S_2 & \end{array} + D + \begin{array}{ccc} S_2 & & S_0 \\ {}_{\beta_2}\diagdown & & \diagup_{g_0} \\ & S_0 & \end{array}$
$r=1$	$\begin{array}{ccc} S_0 & & S_2 \\ {}_{g_0}\diagdown & & \diagup_{\beta_2} \\ & S_0 & \end{array} + D + \begin{array}{ccc} S_0 & & S_1 \\ {}_{\alpha_0}\diagdown & & \diagup_{c_1} \\ & S_0 & \end{array}$
$r=2$	$\begin{array}{ccc} S_1 & & S_0 \\ {}_{c_1}\diagdown & & \diagup_{\alpha_0} \\ & S_0 & \end{array} + D + \begin{array}{ccc} S_0 & & S_1 \\ {}_{\beta_0}\diagdown & & \diagup_{g_1} \\ & S_1 & \end{array}$
$r=3$	$\begin{array}{ccc} S_1 & & S_0 \\ {}_{g_1}\diagdown & & \diagup_{\beta_0} \\ & S_1 & \end{array} + D + \begin{array}{ccc} S_1 & & S_2 \\ {}_{\alpha_1}\diagdown & & \diagup_{c_2} \\ & S_1 & \end{array}$
$r=4$	$\begin{array}{ccc} S_2 & & S_1 \\ {}_{c_2}\diagdown & & \diagup_{\alpha_1} \\ & S_1 & \end{array} + D + \begin{array}{ccc} S_1 & & S_2 \\ {}_{\beta_1}\diagdown & & \diagup_{g_2} \\ & S_2 & \end{array}$
$r=5$	$\begin{array}{ccc} S_2 & & S_1 \\ {}_{g_2}\diagdown & & \diagup_{\beta_1} \\ & S_2 & \end{array} + D + \begin{array}{ccc} S_2 & & S_0 \\ {}_{\alpha_2}\diagdown & & \diagup_{c_0} \\ & S_2 & \end{array}$

Let $B_{\bullet\bullet} = \{B_{ij}, \Delta^{(h)}_{ij} : B_{ij} \to B_{i-1,j}, \Delta^{(v)}_{ij} : B_{ij} \to B_{i,j-1}\}$ be the bicomplex (4.1) lying in the first quadrant of the plane (i.e. $B_{ij} = 0$ if $i < 0$ or $j < 0$), where i denotes the column index and j denotes the row index. For any $l \in \mathbb{Z}$, the bicomplex has the same B_{ij}, $\Delta^{(h)}_{i+1,j}$, $\Delta^{(v)}_{i,j+1}$ (modulo the minus signs) on the diagonal line $j = i + l$ $(i, j \geqslant 0)$. The rows of the bicomplex are periodic to the right, and the columns are periodic to the top, of the diagonal $i = j$, with period 6. The minus signs are put everywhere on the horizontal maps of the odd rows.

$$
\begin{array}{ccccccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
{\scriptstyle\alpha_0}\big\downarrow & & {\scriptstyle\beta_2}\big\downarrow & & {\scriptstyle\alpha_2}\big\downarrow & & {\scriptstyle\beta_1}\big\downarrow & & {\scriptstyle\alpha_1}\big\downarrow & & {\scriptstyle\beta_0}\big\downarrow & & {\scriptstyle\alpha_0}\big\downarrow & & \\
P_0 & \xleftarrow{c_0} & P_2 & \xleftarrow{g_2} & P_2 & \xleftarrow{c_2} & P_1 & \xleftarrow{g_1} & P_1 & \xleftarrow{c_1} & P_0 & \xleftarrow{g_0} & P_0 & \xleftarrow{\beta_0} & \cdots \\
{\scriptstyle\beta_2}\big\downarrow & & {\scriptstyle\alpha_2}\big\downarrow & & {\scriptstyle\beta_1}\big\downarrow & & {\scriptstyle\alpha_1}\big\downarrow & & {\scriptstyle\beta_0}\big\downarrow & & {\scriptstyle\alpha_0}\big\downarrow & & {\scriptstyle c_1}\big\downarrow & & \\
P_2 & \xleftarrow{-g_2} & P_2 & \xleftarrow{-c_2} & P_1 & \xleftarrow{-g_1} & P_1 & \xleftarrow{-c_1} & P_0 & \xleftarrow{-g_0} & P_0 & \xleftarrow{-\beta_0} & P_1 & \xleftarrow{-\alpha_1} & \cdots \\
{\scriptstyle\alpha_2}\big\downarrow & & {\scriptstyle\beta_1}\big\downarrow & & {\scriptstyle\alpha_1}\big\downarrow & & {\scriptstyle\beta_0}\big\downarrow & & {\scriptstyle\alpha_0}\big\downarrow & & {\scriptstyle c_1}\big\downarrow & & {\scriptstyle g_1}\big\downarrow & & \\
P_2 & \xleftarrow{c_2} & P_1 & \xleftarrow{g_1} & P_1 & \xleftarrow{c_1} & P_0 & \xleftarrow{g_0} & P_0 & \xleftarrow{\beta_0} & P_1 & \xleftarrow{\alpha_1} & P_1 & \xleftarrow{\beta_1} & \cdots \\
{\scriptstyle\beta_1}\big\downarrow & & {\scriptstyle\alpha_1}\big\downarrow & & {\scriptstyle\beta_0}\big\downarrow & & {\scriptstyle\alpha_0}\big\downarrow & & {\scriptstyle c_1}\big\downarrow & & {\scriptstyle g_1}\big\downarrow & & {\scriptstyle c_2}\big\downarrow & & \\
P_1 & \xleftarrow{-g_1} & P_1 & \xleftarrow{-c_1} & P_0 & \xleftarrow{-g_0} & P_0 & \xleftarrow{-\beta_0} & P_1 & \xleftarrow{-\alpha_1} & P_1 & \xleftarrow{-\beta_1} & P_2 & \xleftarrow{-\alpha_2} & \cdots \\
{\scriptstyle\alpha_1}\big\downarrow & & {\scriptstyle\beta_0}\big\downarrow & & {\scriptstyle\alpha_0}\big\downarrow & & {\scriptstyle c_1}\big\downarrow & & {\scriptstyle g_1}\big\downarrow & & {\scriptstyle c_2}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & \\
P_1 & \xleftarrow{c_1} & P_0 & \xleftarrow{g_0} & P_0 & \xleftarrow{\beta_0} & P_1 & \xleftarrow{\alpha_1} & P_1 & \xleftarrow{\beta_1} & P_2 & \xleftarrow{\alpha_2} & P_2 & \xleftarrow{\beta_2} & \cdots \\
{\scriptstyle\beta_0}\big\downarrow & & {\scriptstyle\alpha_0}\big\downarrow & & {\scriptstyle c_1}\big\downarrow & & {\scriptstyle g_1}\big\downarrow & & {\scriptstyle c_2}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & {\scriptstyle c_0}\big\downarrow & & \\
P_0 & \xleftarrow{-g_0} & P_0 & \xleftarrow{-\beta_0} & P_1 & \xleftarrow{-\alpha_1} & P_1 & \xleftarrow{-\beta_1} & P_2 & \xleftarrow{-\alpha_2} & P_2 & \xleftarrow{-\beta_2} & P_0 & \xleftarrow{-\alpha_0} & \cdots \\
{\scriptstyle\alpha_0}\big\downarrow & & {\scriptstyle c_1}\big\downarrow & & {\scriptstyle g_1}\big\downarrow & & {\scriptstyle c_2}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & {\scriptstyle c_0}\big\downarrow & & {\scriptstyle g_0}\big\downarrow & & \\
P_0 & \xleftarrow{\beta_0} & P_1 & \xleftarrow{\alpha_1} & P_1 & \xleftarrow{\beta_1} & P_2 & \xleftarrow{\alpha_2} & P_2 & \xleftarrow{\beta_2} & P_0 & \xleftarrow{\alpha_0} & P_0 & \xleftarrow{\beta_0} & \cdots
\end{array}
\tag{4.1}
$$

Proposition 4.2. *The minimal projective resolution of the R-module S_0 coincides with the total complex of the bicomplex $B_{\bullet\bullet}$.*

We emphasize that the main difficulty of this step of our method is not in proving, but in finding the bicomplex, whose periodic properties can be much more complicated for other families of algebras. Although Proposition 4.2 can also be proved by a straightforward verification of exactness or by using a spectral sequence as in [5], our version of the diagrammatic method seems to be the most convenient tool to find the bicomplex.

Corollary 4.3. *Let $m \geqslant 2$ be an integer. Suppose $m = 6q + r$ with $q, r \in \mathbb{Z}$*

and $0 \leqslant r \leqslant 5$. *Then* $Q_m^{(0)} \simeq P_0^n \bigoplus P_1^{n'} \bigoplus P_2^{n''}$ *with*

$$(n, n', n'') = \begin{cases} (2q+1, 2q, 2q), & \textit{if} \ \ r = 0, \\ (2q+1, 2q+1, 2q), & \textit{if} \ \ r = 1, \\ (2q+1, 2q+2, 2q), & \textit{if} \ \ r = 2, \\ (2q+1, 2q+2, 2q+1), & \textit{if} \ \ r = 3, \\ (2q+1, 2q+2, 2q+2), & \textit{if} \ \ r = 4, \\ (2q+2, 2q+2, 2q+2), & \textit{if} \ \ r = 5. \end{cases}$$

The following corollary gives the dimensions of $\mathrm{Ext}_R^m(S_i, S_j)$. The indices of S_i should be considered modulo 3, i.e. $S_3 = S_0$ and $S_4 = S_1$.

Corollary 4.4. *Let* $m \geqslant 2$ *be an integer. Suppose* $m = 6q + r$ *with* $q, r \in \mathbb{Z}$ *and* $0 \leqslant r \leqslant 5$. *Then*

a) $\dim_K \mathrm{Ext}_R^m(S_i, S_i) = \begin{cases} 2q+1, & \textit{if} \ \ r = 0,1,2,3,4, \\ 2q+2, & \textit{if} \ \ r = 5; \end{cases}$

b) $\dim_K \mathrm{Ext}_R^m(S_i, S_{i+1}) = \begin{cases} 2q, & \textit{if} \ \ r = 0, \\ 2q+1, & \textit{if} \ \ r = 1, \\ 2q+2, & \textit{if} \ \ r = 2,3,4,5; \end{cases}$

c) $\dim_K \mathrm{Ext}_R^m(S_i, S_{i+2}) = \begin{cases} 2q, & \textit{if} \ \ r = 0,1,2, \\ 2q+1, & \textit{if} \ \ r = 3, \\ 2q+2, & \textit{if} \ \ r = 4,5. \end{cases}$

Remark 4.5. By Proposition 4.2, we have $Q_m^{(0)} = \bigoplus_{i+j=m} B_{ij}$. The modules in this direct sum will be always ordered with respect to the first index, for example, we write $Q_3^{(0)} = B_{03} \oplus B_{12} \oplus B_{21} \oplus B_{30} = P_1 \oplus P_0 \oplus P_1 \oplus P_2$. The simple direct summands of $\operatorname{top} \Omega^m(S_i) \simeq \operatorname{top} Q_m^{(i)}$ will be ordered in the same way: $\operatorname{top} Q_3^{(0)} = S_1 \oplus S_0 \oplus S_1 \oplus S_2$. We call such decompositions of $Q_m^{(i)}$ and $\operatorname{top} Q_m^{(i)}$ the *canonical* decompositions.

5 Generators

In this section, we indicate a finite set of generators for the Yoneda algebra:

$$\mathcal{Y}(R) = \mathcal{E}(R/J(R)) = \bigoplus_{m \geqslant 0} \bigoplus_{i,j=0}^{2} \mathrm{Ext}_R^m(S_i, S_j).$$

Let us recall some facts and notation related to the Yoneda algebra (see also [13, Chapter 2]). Since S_j is a simple R-module, we have $\mathrm{Ext}^m_R(S_i, S_j) \simeq \mathrm{Hom}_R(\Omega^m(S_i), S_j)$. Let ψ be an element of $\mathrm{Ext}^m_R(S_i, S_j)$. Its image $\widehat{\psi}$ in $\mathrm{Hom}_R(\Omega^m(S_i), S_j)$ induces a morphism of projective resolutions $\{f_l : Q^{(i)}_{m+l-1} \to Q^{(j)}_{l-1} \,|\, l \geqslant 1\}$ and a homomorphism $f_0 : Q^{(i)}_{m-1} \to P_j$. We have a commutative diagram:

$$\begin{array}{ccccc} Q^{(i)}_m & \xrightarrow{d^{(i)}_{m-1}} & \Omega^m(S_i) & \subset & Q^{(i)}_{m-1} \\ \downarrow f_1 & & \downarrow \widehat{\psi} & & \downarrow f_0 \\ Q^{(j)}_0 & \xrightarrow{d^{(j)}_{-1}} & S_j & \subset & P_j \end{array} . \tag{5.1}$$

We see that $\widehat{\psi}$ can be represented by the outer square of (5.1) because this commutative square uniquely defines the map $\widehat{\psi}$. Moreover, $\widehat{\psi}$ is uniquely defined by providing only a homomorphism $f_1 : Q^{(i)}_m \to Q^{(j)}_0$ such that $d^{(j)}_{-1} f_1$ annihilates $\mathrm{Ker}\, d^{(i)}_{m-1}$. In this case we write $\widehat{\psi} = \mathrm{sq}\big(Q^{(i)}_m \xrightarrow{f_1} Q^{(j)}_0\big)$. The homomorphisms

$$\Omega^l(\widehat{\psi}) : \Omega^{m+l}(S_i) \to \Omega^l(S_j), \quad \Omega^l(\widehat{\psi}) = f_l|_{\Omega^{m+l}(S_i)},$$

are called *the* Ω*-translates of* $\widehat{\psi}$. We have $\Omega^l(\widehat{\psi}) = \mathrm{sq}\big(Q^{(i)}_{m+l} \xrightarrow{f_{l+1}} Q^{(j)}_l\big)$. If $\varphi \in \mathrm{Ext}^n_R(S_j, S_e) \simeq \mathrm{Hom}_R(\Omega^n(S_j), S_e)$, the Yoneda product $\varphi\psi \in \mathrm{Ext}^{m+n}_R(S_i, S_e)$ has the image $\widehat{\varphi\psi} = \widehat{\varphi} \cdot \Omega^n(\widehat{\psi})$ in $\mathrm{Hom}_R(\Omega^{m+n}(S_i), S_e)$. Moreover, if $\widehat{\varphi} = \mathrm{sq}\big(Q^{(j)}_n \xrightarrow{g} Q^{(e)}_0\big)$, then $\widehat{\varphi\psi} = \mathrm{sq}\big(Q^{(i)}_{m+n} \xrightarrow{gf_{n+1}} Q^{(e)}_0\big)$. Although the maps f_l and the Ω-translates are not uniquely determined by $\widehat{\psi}$, it is easily seen that the resulting map to a simple module does not depend on their choice. Since R is a QF-algebra, we can also translate the maps from left to right: any map $\rho : \Omega^{m+l}(S_i) \to \Omega^l(S_j)$ induces a map $\tilde{\rho} : \Omega^m(S_i) \to S_j$ such that $\rho = \Omega^l(\tilde{\rho})$.

Consider the homogeneous elements of $\mathcal{Y}(R)$ defined as follows:

$$x_i \in \mathrm{Ext}^1_R(S_i, S_i),\ y_i \in \mathrm{Ext}^2_R(S_i, S_i),\ z_i \in \mathrm{Ext}^1_R(S_i, S_{i+1}),\ i = 0, 1, 2,$$
$$\widehat{x}_i = \mathrm{sq}(Q^{(i)}_1 \xrightarrow{(1,0)} Q^{(i)}_0),\ \widehat{y}_i = \mathrm{sq}(Q^{(i)}_2 \xrightarrow{(0,-1,0)} Q^{(i)}_0),$$
$$\widehat{z}_i = \mathrm{sq}(Q^{(i)}_1 \xrightarrow{(0,1)} Q^{(i+1)}_0),$$

where the index $i+1$ is considered modulo 3. It will cause no confusion to use the same letters as for the elements of $\mathcal{E}$ (we will use only $x_i, y_i, z_i \in \mathcal{Y}$ in this section and only $x_i, y_i, z_i \in \mathcal{E}$ in the proof of Proposition 6.1). To show

how we compute the Ω-translates, let us determine $\Omega^1(\widehat{x}_0)$. The map $\widehat{x}_0$ is defined by the right square in the diagram

$$\begin{array}{ccccc}
Q_2^{(0)} & \xrightarrow{\begin{pmatrix}\beta_0 & -g_0 & 0\\ 0 & c_1 & \alpha_1\end{pmatrix}} & Q_1^{(0)} & \xrightarrow{(\alpha_0,\beta_0)} & Q_0^{(0)} \\
U\big\downarrow ? & & (1,0)\big\downarrow & & g_0\big\downarrow \\
Q_1^{(0)} & \xrightarrow[(\alpha_0,\beta_0)]{} & Q_0^{(0)} & \xrightarrow[\alpha_0^s]{} & P_0\ .
\end{array}$$

We have to find a map $U : P_1 \bigoplus P_0 \bigoplus P_1 \rightarrow P_1 \bigoplus P_0$ such that the diagram commutes, and therefore $\Omega^1(\widehat{x}_0) = \mathrm{sq}(Q_2^{(0)} \xrightarrow{U} Q_1^{(0)})$. Writing the corresponding matrix equation, we see that we can take, for example,

$$U = \begin{pmatrix} 0 & -\alpha_0^{s-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

Proposition 5.1. *The extension groups below have the following K-bases:*

$$\begin{array}{ll}
\mathrm{Ext}_R^1(S_i, S_i) = \langle x_i \rangle, & \mathrm{Ext}_R^1(S_i, S_{i+1}) = \langle z_i \rangle, \\
\mathrm{Ext}_R^2(S_i, S_i) = \langle y_i \rangle, & \mathrm{Ext}_R^2(S_i, S_{i+1}) = \langle z_i x_i, x_{i+1} z_i \rangle.
\end{array}$$

Proof. The result for the first three groups is clear by Corollary 4.4 because the elements x_i, y_i, z_i are nonzero. We prove $\mathrm{Ext}_R^2(S_0, S_1) = \langle z_0 x_0, x_1 z_0 \rangle$.

Since $\widehat{z}_0 = \mathrm{sq}(Q_1^{(0)} \xrightarrow{(0,1)} Q_0^{(1)})$ and $\Omega^1(\widehat{x}_0) = \mathrm{sq}(Q_2^{(0)} \xrightarrow{U} Q_1^{(0)})$ with (5.2), we have $\widehat{z_0 x_0} = \widehat{z}_0 \cdot \Omega^1(\widehat{x}_0) = \mathrm{sq}(Q_2^{(0)} \xrightarrow{(0,1)U} Q_0^{(1)}) = \mathrm{sq}(Q_2^{(0)} \xrightarrow{(1,0,0)} Q_0^{(1)})$. In the same manner we obtain $\widehat{x_1 z_0} = \widehat{x}_1 \cdot \Omega^1(\widehat{z}_0) = \mathrm{sq}(Q_2^{(0)} \xrightarrow{(0,0,1)} Q_0^{(1)})$. We see now that $z_0 x_0$ and $x_1 z_0$ are linearly independent. It remains to note that $\dim_K \mathrm{Ext}_R^2(S_0, S_1) = 2$ by Corollary 4.4. □

Lemma 5.2. *Let S be a simple R-module, M_1 and M_2 be two R-modules with diagrams D_1 and D_2 with respect to the bases $\{v_{1p}\}_p \subset M_1$ and $\{v_{2q}\}_q \subset M_2$ respectively. Suppose that D' is a closed subdiagram of D_1 and an open subdiagram of D_2. Let l be a common vertex of D'_{top} and D_{2top}. Suppose that $f : M_1 \rightarrow S$ is an R-homomorphism such that the $f(v_{1j}) = 0$ for any vertex $j \in D_{1top}$, $j \neq l$. Then there exist R-homomorphisms $\rho : M_1 \rightarrow M_2$ and $f' : M_2 \rightarrow S$ such that $f = f'\rho$.*

Proof. Let M' be the submodule of M_2 defined by D', we identify it with the corresponding to D' quotient of M_1. Let $\pi : M_1 \rightarrow M'$ and $i : M' \rightarrow M_2$ be the canonical epimorphism and monomorphism respectively. Set $\rho = i\pi$. We have $\rho(v_{1l}) = i(\pi(v_{1l})) = i(v_{2l}) = v_{2l}$ and $l \in D_{1top}$ since $D'_{top} \subset D_{1top}$.

As S is a simple R-module, for any $h \in \mathrm{Hom}_R(M_i, S)$ we have $h(\mathrm{Rad}\, M_i) = 0$, hence $\mathrm{Hom}_R(M_i, S) \simeq \mathrm{Hom}_R(\mathrm{top}\, M_i, S)$. Denote by $\overline{h} \in \mathrm{Hom}_R(\mathrm{top}\, M_i, S)$ the image of h. Since $\mathrm{top}\, M_2 = \bigoplus_{j \in D_{2\mathrm{top}}} K\overline{v_{2j}}$, we can define a homomorphism $\overline{f'} : \mathrm{top}\, M_2 \to S$ by $\overline{f'}(\overline{v_{2l}}) = f(v_{1l})$ and $\overline{f'}(\overline{v_{2j}}) = 0$ for any $j \in D_{2top}$, $j \neq l$. It is easily seen that the corresponding $f' \in \mathrm{Hom}_R(M_2, S)$ satisfies $f = f'\rho$. □

Proposition 5.3. *The set $\mathcal{X} = \{x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2\}$ generates the Yoneda algebra $\mathcal{Y}(R)$ as a K-algebra.*

Proof. We prove by induction on m that the groups $\mathrm{Ext}_R^m(S_i, S_j)$ are generated by some products of elements of $\mathcal{X}$. For $m \leqslant 2$, this follows directly from Proposition 5.1 and Corollary 4.4. Assume that $m \geqslant 3$ and that our statement holds for all $\mathrm{Ext}_R^{m'}(S_i, S_j)$ with $m' < m$, we will prove it for m.

Using the isomorphism $\mathrm{Ext}_R^m(S_i, S_j) \simeq \mathrm{Hom}_R(\Omega^m(S_i), S_j)$, we represent an element of the group $\mathrm{Ext}_R^m(S_i, S_j)$ by the corresponding map $f \colon \Omega^m(S_i) \to S_j$. Since $\mathrm{Hom}_R(\Omega^m(S_i), S_j) \simeq \mathrm{Hom}_R(\mathrm{top}\,(\Omega^m(S_i)), S_j)$ and $\mathrm{top}\,(\Omega^m(S_i))$ is a direct sum of simple modules, we can assume without loss of generality that f induces a nonzero map on at most one simple direct summand in the canonical decomposition of $\mathrm{top}\,\Omega^m(S_i)$ (see Remark 4.5).

a) Assume that $f \colon \Omega^m(S_i) \to S_j$ induces zero maps on the extreme (the left and the right) simple direct summands of $\mathrm{top}\,\Omega^m(S_i)$. It follows from Proposition 4.1 that the diagram of $\Omega^{m-2}(S_i)$ is a closed subdiagram in that of $\Omega^m(S_i)$, hence $\Omega^{m-2}(S_i)$ is a quotient of $\Omega^m(S_i)$. Applying Lemma 5.2 with $M_1 = \Omega^m(S_i)$ and $M_2 = M' = \Omega^{m-2}(S_i)$, we have $f = f'\rho$ for some $\rho \in \mathrm{Hom}_R(\Omega^m(S_i), \Omega^{m-2}(S_i))$ and $f' \in \mathrm{Hom}_R(\Omega^{m-2}(S_i), S_j)$. Since $\rho = \Omega^{m-2}(\tilde{\rho})$ for some homomorphism $\tilde{\rho} : \Omega^2(S_i) \to S_i$, the desired statement follows from $f = f' {\cdot} \Omega^{m-2}(\tilde{\rho})$ and the induction hypothesis for $f' \in \mathrm{Hom}_R(\Omega^{m-2}(S_i), S_j) \simeq \mathrm{Ext}_R^{m-2}(S_i, S_j)$ and $\tilde{\rho} \in \mathrm{Hom}_R(\Omega^2(S_i), S_i) \simeq \mathrm{Ext}_R^2(S_i, S_i)$.

b) Assume now that f induces a nonzero map on an extreme direct summand of $\mathrm{top}\,\Omega^m(S_i)$. The proof is a straightforward verification of several similar cases. In each case we can find $M_2 = \Omega^1(S_l)$ and $M' \subset M_2$ such that Lemma 5.2 applies for $M_1 = \Omega^m(S_i)$, M_2 and M'. We consider in detail only one case: $i = 0$, $j = 2$, $m \equiv 4 \,(\mathrm{mod}\, 6)$, $f \colon \Omega^m(S_0) \to S_2$ induces a nonzero map only on the extreme left direct summand in $\mathrm{top}\,\Omega^m(S_0)$. The proof of the other cases is left to the reader.

Let D_1 be the diagram of $M_1 = \Omega^m(S_0)$. Define D_2 and D' as follows:

$$D_2 : \quad \begin{array}{ccccc} S_1 & & & & S_2 \\ & {}_{g_1}\diagdown & & \diagup_{c_2} & \\ & & S_1 & & \end{array}, \qquad D' : \quad \begin{array}{ccccc} S_2 & & & & S_1 \\ & {}_{c_2}\diagdown & & \diagup_{\alpha_1} & \\ & & S_1 & & \end{array}.$$

We see that D_2 is the diagram of $M_2 = \Omega^1(S_1)$ and D' is an open subdiagram of D_2 (we have $t \geqslant 2$ and $g_1 = \alpha_1^{t-2}\alpha_1$). By Proposition 4.1, D' is a closed subdiagram of D_1. We see that f induces a nonzero map on the direct summand corresponding to the extreme left vertex of $D_{1\text{top}}$ and D'_{top}, and that this vertex belongs also to $D_{2\text{top}}$ (where it becomes the right one with respect to the canonical decomposition). By Lemma 5.2 we have $f = f'\rho$ for some $\rho \in \mathrm{Hom}_R(\Omega^m(S_0), \Omega^1(S_1))$ and $f' \in \mathrm{Hom}_R(\Omega^1(S_1), S_2)$. Since $\rho = \Omega^1(\tilde{\rho})$ for some homomorphism $\tilde{\rho} : \Omega^{m-1}(S_0) \to S_1$, our statement follows from $f = f'{\cdot}\Omega^1(\tilde{\rho})$ and the induction hypothesis for $f' \in \mathrm{Hom}_R(\Omega^1(S_1), S_2) \simeq \mathrm{Ext}^1_R(S_1, S_2)$ and $\tilde{\rho} \in \mathrm{Hom}_R(\Omega^{m-1}(S_0), S_1) \simeq \mathrm{Ext}^{m-1}_R(S_0, S_1)$. □

Proposition 5.4. *The elements of $\mathcal{X} \subset \mathcal{Y}(R)$ satisfy the relations* (2.2).

Proof. We prove only $x_0^2 = \delta(s, 2)y_0$. The verification of the other relations is similar and is left to the reader.

Since $\widehat{x}_0 = \mathrm{sq}(Q_1^{(0)} \xrightarrow{(1,0)} Q_0^{(0)})$ and $\Omega^1(\widehat{x}_0) = \mathrm{sq}(Q_2^{(0)} \xrightarrow{U} Q_1^{(0)})$ with (5.2), we have $\widehat{(x_0^2)} = \widehat{x}_0 \cdot \Omega^1(\widehat{x}_0) = \mathrm{sq}(Q_2^{(0)} \xrightarrow{(1,0)U} Q_0^{(1)}) = \mathrm{sq}(Q_2^{(0)} \xrightarrow{(0,-\alpha_0^{s-2},0)} Q_0^{(0)})$. If $s > 2$, this map obviously induces a zero map $\Omega^2(S_0) \to S_0$, which implies $x_0^2 = 0$. If $s = 2$, $\widehat{(x_0^2)}$ coincides with $\widehat{y}_0$, therefore $x_0^2 = y_0$. □

6 Proof of Theorem 2.1

Let $\mathcal{E} = \bigoplus_{m\geqslant 0} \mathcal{E}^m$ and $\mathcal{Y} = \bigoplus_{m\geqslant 0} \mathcal{Y}^m$ be the decompositions of $\mathcal{E}$ and $\mathcal{Y}$ into homogeneous direct summands. Let ε_i denote the idempotents of $K[T]$ corresponding to the vertices $i = 0, 1, 2$ of T as well as their images in $\mathcal{E}$. We use the same notation for the idempotents $\varepsilon_i = \mathrm{Id}_{S_i} \in \mathcal{Y}$.

By Propositions 5.3 and 5.4, there exists an epimorphism of graded K-algebras $\varphi : \mathcal{E} \longrightarrow \mathcal{Y}$ with $\varphi(\varepsilon_i) = \varepsilon_i$, $\varphi(x_i) = x_i$, $\varphi(y_i) = y_i$, $\varphi(z_i) = z_i$. To prove Theorem 2.1 it remains to show that φ is a monomorphism. It follows from the following result.

Proposition 6.1. *For any $m \geqslant 0$, we have* $\dim_K \mathcal{E}^m = \dim_K \mathcal{Y}^m$.

Proof. It is sufficient to prove that for any $k \in \{0, 1, 2\}$ and $m \geqslant 0$,

$$\dim_K(\mathcal{E}^m\varepsilon_k) = \dim_K(\mathcal{Y}^m\varepsilon_k). \tag{6.1}$$

We prove it for $k = 0$.

Note that we can omit the indices of x_i, y_i, z_i in a nonzero path in $K[T]$ starting in 0, because such a path is uniquely determined by the sequence of letters. The indices can be written in a unique way from right to left.

To shorten notation in this proof, we will not write indices in the nonzero paths of $K[T]\varepsilon_0$ as well as in their images in $\mathcal{E}\varepsilon_0$. For example, the path $x_2z_1x_1z_0x_0z_2x_2z_1x_1z_0x_0y_0^4$ will be briefly denoted by $xzxzxzxzxzxy^4$ or even by $(xz)^5xy^4$.

It follows easily from (2.2) that the monomials

$$(zx)^j y^i,\ (xz)^j xy^i,\ (zx)^j zy^i,\ (xz)^{j+1} y^i,\quad i,j \geqslant 0, \tag{6.2}$$

form a K-basis of $\mathcal{E}\varepsilon_0$. The monomials of degree m in (6.2) form a K-basis of $\mathcal{E}^m\varepsilon_0$.

For $m \leqslant 5$ the relations (6.1) are verified directly. Let us assume that $m \geqslant 6$. We claim that $\dim_K(\mathcal{E}^m\varepsilon_0) = \dim_K(\mathcal{E}^{m-6}\varepsilon_0) + 6$. Indeed, the basis elements of $\mathcal{E}^{m-6}\varepsilon_0$ are in one-to-one correspondence with those basis elements of $\mathcal{E}^m\varepsilon_0$ for which $i \geqslant 3$ (this correspondence is given by replacing i by $i+3$). There are exactly six basis elements in $\mathcal{E}^m\varepsilon_0$ with $i \leqslant 2$, these elements are $\{(xz)^{(m-2i-1)/2}xy^i,\ (zx)^{(m-2i-1)/2}zy^i \mid i = 0,1,2\}$ if m is odd, and $\{(zx)^{(m-2i)/2}y^i,\ (xz)^{(m-2i)/2}y^i \mid i = 0,1,2\}$ if m is even, which completes the proof of our claim.

On the other hand, $\dim_K(\mathcal{Y}^m\varepsilon_k) = \dim_K(\mathcal{Y}^{m-6}\varepsilon_k) + 6$ by Corollary 4.4 since $\mathcal{Y}^m\varepsilon_k = \mathrm{Ext}_R^m(S_k, \overline{R})$. The formula (6.1) follows by induction on m. □

References

[1] K. Erdmann, Blocks of tame representation type and related algebras, Lecture Notes in Math. Vol. 1428, Springer-Verlag, Berlin, 1990.

[2] A.I.Generalov, Cohomology of algebras of dihedral type, I, Zapiski nauchn. sem. POMI 265(1999), 139–162 (in Russian); English transl. in: J. Math. Sci. 112(2002), no. 3, 4318–4331.

[3] O.I.Balashov, A.I.Generalov, The Yoneda algebras for some class of dihedral algebras, Vestnik St.Peterburg Univ. Ser. 1, 3(1999), no. 15, 3–10 (in Russian); English transl. in: Vestnik St.Peterburg Univ. Math. 32(1999), no. 3, 1–8.

[4] O.I.Balashov, A.I.Generalov, Cohomology of algebras of dihedral type, II, Algebra i analiz 13(2001), no. 1, 3–25 (in Russian); English transl. in: St.Petersburg Math. J. 13(2002), no.1, 1–16.

[5] A.I.Generalov, Cohomology of algebras of semidihedral type, I, Algebra i analiz, 13(2001), 4, 54–85 (in Russian); English transl. in: St. Petersburg Math. J. 13(2002), no.4, 549–573.

[6] M.A.Antipov, A.I.Generalov, Cohomology of algebras of semidihedral type, II, Zapiski nauchn. sem. POMI, 289(2002) 9–36 (in Russian); English transl. in: J. Math. Sci. (to appear).

[7] A.I.Generalov, E.A.Osiuk, Cohomology of algebras of dihedral type, III: the family $D(2\mathcal{A})$, Zapiski nauchn. sem. POMI, 289(2002), 113–133 (in Russian); English transl. in: J. Math. Sci. (to appear).

[8] A.I.Generalov, Cohomology of algebras of dihedral type, IV: the family $D(2\mathcal{B})$, Zapiski nauchn. sem. POMI, 289(2002) 76–89 (in Russian); English transl. in J. Math. Sci. (to appear).

[9] N.Kosmatov, Resolut, http://www.lifc.univ-fcomte.fr/∼kosmatov/resolut/, 2003.

[10] D.J.Benson, J.F.Carlson, Diagrammatic methods for modular representations and cohomology, Commun. in Algebra 15(1987), no. 1–2, 53–121.

[11] J.L.Alperin, Diagrams for modules, J. Pure Appl. Algebra 16 (1980), no. 2, 111–119.

[12] K.R.Fuller, Algebras from diagrams, J. Pure Appl. Algebra 48 (1987), no. 1–2, 23–37.

[13] D.J.Benson, Representations and cohomology, I: Basic representation theory of finite groups and associative algebras, Cambridge Univ. Press, Cambridge, 1991.