Projective Resolutions and Yoneda Algebras for Algebras of Dihedral Type: the family $D(3\mathcal{Q})$.

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Abstract

This paper provides a method for the computation of Yoneda algebras for algebras of dihedral type. The Yoneda algebras for one infinite family of algebras of dihedral type (the family $D(3\mathcal{Q})$ in K. Erdmann's notation) are computed. The minimal projective resolutions of simple modules were calculated by an original computer program implemented by one of the authors in `C++` language. The algorithm of the program is based on a diagrammatic method presented in this paper and inspired by that of D. Benson and J. Carlson.

Keywords: Yoneda algebra, algebras of dihedral type, projective resolutions, module diagrams.

1 Introduction

The algebras of dihedral, semidihedral and quaternion type were defined and classified by Karin Erdmann in [1]. They generalize the blocks with dihedral, semidihedral and generalized quaternion defect groups respectively. The classification contains dozens of infinite families of algebras. Each family is defined by a quiver with relations containing some parameters.

The Yoneda algebras of some algebras of dihedral and semidihedral types were computed by the first author et al. in [2]–[10]. This computation contains two steps: to find the projective resolutions of simple modules, and to determine the Yoneda algebra. For the algebras that appear as principal blocks of group algebras, these results allowed to find the cohomology ring of the corresponding groups.

It turns out that for all considered algebras, the minimal projective resolution of a simple module can be represented as the total complex of an infinite bicomplex (except the cases where this module is Ω-periodic). The bicomplex repeats itself in some regular way. To understand the periodic properties of the bicomplex, it is often necessary to determine its first 10–20 diagonals. This computation being rather difficult to do by hand, the object of this work is not only to find the Yoneda algebras for other families of dihedral algebras, but also to use computer-based techniques to find the projective resolutions.

Recently [10] we have already applied our method to determine the Yoneda algebra for one infinite family of dihedral algebras: the family $D(3\mathcal{L})$ in the notation of [1]. The projective resolutions for this family were computed by an original `C++` program *Resolut* [11] implemented by the second author. In this paper, we give another application of our technique and compute the Yoneda algebra for algebras that constitute the family $D(3\mathcal{Q})$.

The algorithm of the program is based on a diagrammatic method inspired by that of David Benson and Jon Carlson [12]. Although our definition of a diagram is different from those of [12, 13, 14], many ideas and diagram constructions of [12] still apply in our case. An important advantage of our approach is the possibility to implement a significant part of the Yoneda algebra computation in a computer program.

The program *Resolut* examines the algebra defined by the given quiver with relations (with fixed values of parameters) and computes the minimal projective resolutions of the simple modules over this algebra. It takes less than one second to compute sufficiently many modules in the bicomplex to see its structure. Running the program for different parameters allows to conjecture the general form of the bicomplex for arbitrary parameters. The conjecture is easy to prove by hand, as the bicomplex contains only finitely many different squares.

The paper is organized as follows. In Section 2, we define the family $D(3\mathcal{Q})$ of dihedral algebras, state our main result and describe our method of computation of Yoneda algebras. Section 3 introduces the notion of a diagram and provides some properties of diagrams. In Section 4, we apply the diagrammatic method to compute the minimal projective resolutions and syzygies of simple modules. We define the generators of the Yoneda algebra in Section 5 and complete the proof of our main result in Section 6.

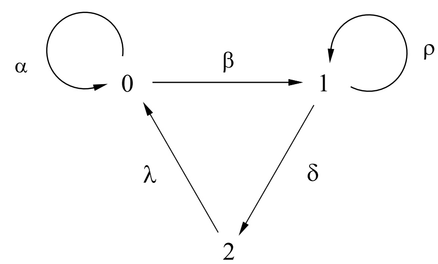
2 Main Result

Let K be a field, Λ be an associative K-algebra with identity, M be a Λ-module (all the considered modules are left modules). The K-module $\mathcal{E}xt(M) = \bigoplus_{m\geqslant 0} \mathrm{Ext}^m_\Lambda(M,M)$ can be endowed with the structure of an associative K-algebra using the Yoneda product [15]. The algebra $\mathcal{E}xt(M)$ is called *the* Ext-*algebra of* M.

If Λ is a basic finite dimensional K-algebra, we set $\overline{\Lambda} = \Lambda/J(\Lambda)$ where $J(\Lambda)$ is the Jacobson radical of Λ. The Ext-algebra $\mathcal{E}xt(\overline{\Lambda})$ is called *the Yoneda algebra of* Λ and is denoted by $\mathcal{Y}(\Lambda)$.

Let k,s,t be integers such that $k \geqslant 1$ and $s,t \geqslant 2$. We define the K-algebra $R_{k,s,t}$ by the following quiver with relations (we write down a composition from the right to the left):

T:

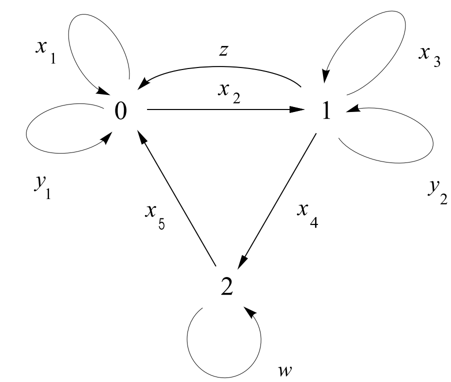


,

$$\begin{gathered} \beta\alpha = \alpha\lambda = \rho\beta = \delta\rho = 0, \\ (\lambda\delta\beta)^k = \alpha^s, \quad (\beta\lambda\delta)^k = \rho^t. \end{gathered} \tag{2.1}$$

The algebras $R_{k,s,t}$ compose an infinite family of dihedral algebras, which is denoted in [1] by $D(3\mathcal{Q})$. Every $R_{k,s,t}$ is a symmetric algebra (and therefore a QF-algebra). To describe the Yoneda algebra $\mathcal{Y}(R_{k,s,t})$, let us consider the following quiver

$\mathcal{R}$:



.

Let $K[\mathcal{R}]$ be the path algebra of $\mathcal{R}$. We define the following grading on $K[\mathcal{R}]$:

$$\deg(x_i) = 1,\ i = 1,2,3,4,5;$$
$$\deg(y_i) = 2,\ i = 1,2;\ \deg(z) = 3,\ \deg w = 6.$$

Consider the following relations on the quiver $\mathcal{R}$:

$$\left.\begin{array}{c} x_1^2 = \delta(s,2)y_1, \quad x_3^2 = \delta(t,2)y_2, \\ x_4x_2 = x_5x_4 = x_2x_5 = x_4y_2 = y_1x_5 = 0, \\ x_1y_1 = y_1x_1,\ y_2x_2 = x_2y_1,\ x_3y_2 = y_2x_3, \\ zx_2 = -\delta(k,1)y_1^2, \quad x_2z = -\delta(k,1)y_2^2, \\ y_1z = zy_2,\ x_4x_3x_2x_1z = -wx_4,\ zx_3x_2x_1x_5 = -x_5w. \end{array}\right\} \tag{2.2}$$

Here $\delta(i,j)$ denotes the Kroneker delta function: $\delta(i,j) = 1$ if $i = j$, and 0 otherwise.

Let $\mathcal{E}_{k,s,t}$ be the K-algebra defined by the quiver $\mathcal{R}$ with the relations (2.2). As all these relations are homogeneous, the algebra $\mathcal{E}_{k,s,t}$ inherits a grading from $K[\mathcal{R}]$. We can now state our main result.

Theorem 2.1. *The Yoneda algebra $\mathcal{Y}(R_{k,s,t})$ is isomorphic, as a graded algebra, to $\mathcal{E}_{k,s,t}$.*

To simplify notation, we set $R = R_{k,s,t}$, $\mathcal{E} = \mathcal{E}_{k,s,t}$ and $\mathcal{Y} = \mathcal{Y}(R)$. We denote by e_i the idempotents of R corresponding to the vertices $i = 0,1,2$ of T. There exist three indecomposable projective R-modules and three simple R-modules (up to isomorphism), they are defined by $P_i = Re_i$ and $S_i = P_i/(J(R)P_i)$, respectively.

Let us now describe our method of computation of the Yoneda algebras. This method can be also applied to other families of dihedral algebras defined in K. Erdmann's classification [1].

1) We examine the given quiver with relations T to find the bases and the diagrams of the indecomposable projective modules.

2) Using the diagrammatic method, we compute the bicomplexes such that their total complexes give minimal projective resolutions of simple modules. We also describe the syzygies in terms of diagrams. The first two steps can be done today by the program *Resolut* [11].
3) We chose some generators in the groups $\mathrm{Ext}^1_R(S_i, S_j)$ and check if they generate the groups $\mathrm{Ext}^2_R(S_i, S_j)$ in the Yoneda algebra. If not, we chose additional generators in $\mathrm{Ext}^2_R(S_i, S_j)$ and so on, until the generators seem to generate the Yoneda algebra.
4) Computing the products of the generators, we find the relations and conjecture a quiver with relations defining the Yoneda algebra. The conjecture is proved as it is shown below.

3 Diagrams

Let Λ be an algebra defined by a quiver with relations W, and let L be the set of edges of W. Let M be a Λ-module and $D = (V, E, \varphi)$ be a finite directed graph with vertices V, edges E and a labelling function $\varphi\colon E \to L$. If $i, j \in V$ and $e \in E$ is an edge $i \to j$ with the label $\varphi(e) = \gamma \in L$, we write $e = e(i, j)$ or $e = e(i, j, \gamma)$.

Definition 3.1. We say that M *has a diagram* D if there exists a K-basis $\{v_i \,|\, i \in V\}$ of M such that
(i) for any edge $e(i, j, \gamma)$, we have $\gamma v_i = v_j$ or $\gamma v_i = -v_j$,
(ii) for any $i \in V$ and $\gamma \in L$ with $\gamma v_i \neq 0$, there exists a unique $j \in V$ such that $e(i, j, \gamma) \in E$,
(iii) for any v_i, the R-module $\mathrm{top}\,(Rv_i)$ is simple, i.e. Rv_i is a local module.

The same module M can have different diagrams according to this definition. As we consider the diagrams with respect to some fixed bases, we do not need the diagram uniqueness in our results. For simplicity of notation, we assume that a non-directed edge in a diagram denotes an arrow from the higher vertex to the lower one, and we write sometimes just $i \in D$ instead of $i \in V$. It is convenient to write the simple module $\mathrm{top}\,(Rv_i)$ in the vertex i of the diagram.

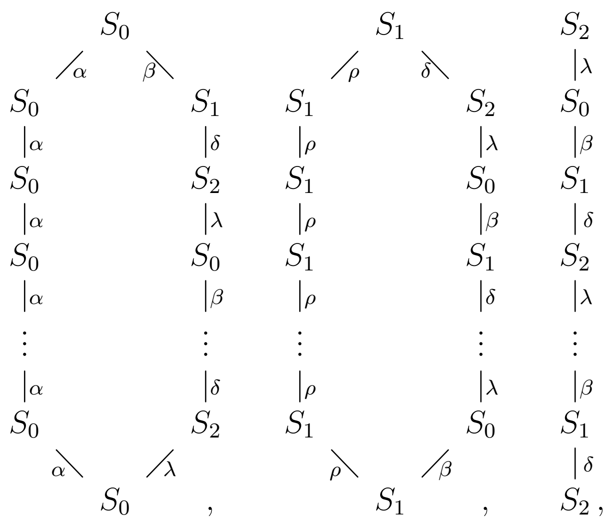
To give an example of diagrams, let us determine the diagrams of the R-modules $P_i = Re_i$, $i = 0, 1, 2$. It is easily seen from (2.1) that P_i have the following K-bases:

$$\begin{aligned} P_0 = \langle\, & e_0,\, \alpha,\, \alpha^2, \ldots,\, \alpha^{s-1},\, \beta,\, \delta\beta,\, \lambda\delta\beta, \\ & \beta\delta\lambda\beta, \ldots,\, \delta\beta(\lambda\delta\beta)^{k-1}, \alpha^s = (\lambda\delta\beta)^k \,\rangle; \end{aligned} \tag{3.1}$$

$$\begin{aligned} P_1 = \langle\, & e_1,\, \rho,\, \rho^2, \ldots,\, \rho^{t-1}, \delta,\, \lambda\delta,\, \beta\lambda\delta, \\ & \delta\beta\lambda\delta, \ldots,\, \lambda\delta(\beta\lambda\delta)^{k-1}, \rho^t = (\beta\lambda\delta)^k \,\rangle; \end{aligned} \tag{3.2}$$

$$P_2 = \langle\, e_2,\, \lambda,\, \beta\lambda,\, \delta\beta\lambda,\, \lambda\delta\beta\lambda, \ldots,\, \beta\lambda(\delta\beta\lambda)^{k-1}, (\delta\beta\lambda)^k \,\rangle. \tag{3.3}$$

We obtain using (3.1)–(3.3) that the modules P_0, P_1 and P_2 have the diagrams



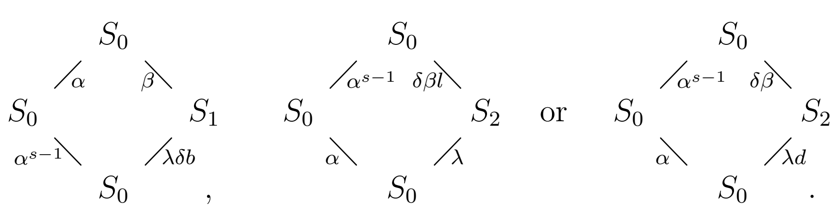
(3.4)

respectively.

Set

$$b = (\beta\lambda\delta)^{k-1},\ d = (\delta\beta\lambda)^{k-1},\ l = (\lambda\delta\beta)^{k-1}$$

(if $k = 1$, we take $b = e_1$, $d = e_2$, $l = e_0$). We will use the same letters for the elements of the path algebra $K[T]$ and for their images in R. For abbreviation, we denote a sequence of edges in a diagram by one edge and write the composition of the original edges nearby. The diagram of P_0 can be written in this notation, for example, as



Although our definition of a diagram is different from that of [12], many definitions and diagrammatic constructions from [12] are applicable in our context. We briefly discuss some definitions and properties which will be useful below. Let $D = (V, E, \varphi)$ be a diagram of M and let $\{v_i \,|\, i \in V\}$ be the corresponding basis of M. If another R-module M' has the same diagram D, then $M \simeq M'$.

We say that $D' = (V', E', \varphi')$ is *a subdiagram of* D if $V' \subset V$, $E' = \{e(i,j) \in E \,|\, i, j \in V'\}$ and $\varphi' = \varphi|_{E'}$. Note that a subdiagram D' of D contains all edges that connect two vertices of D', therefore D' is entirely determined by its set of vertices V'. The R-submodule generated by $\{v_j \,|\, j \in V'\}$ has a diagram which is the subdiagram of D containing all vertices (and therefore all edges) lying on the paths with origine in V'.

We say that a subdiagram D' of D is *open* if for any vertex $j \in D'$, D' contains all vertices lying on the paths in D with origine j. In this case, the R-submodule $M' \subset M$ generated by $\{v_j \,|\, j \in D'\}$ is equal to $\sum_{j \in D'} K v_j$ and has the diagram D'.

Dually, we say that a subdiagram D' of D is *closed* if for any vertex $j \in D'$, D' contains all vertices lying on the paths in D with end j. Let M_0 denote the submodule M_0 generated by $\{v_j \,|\, j \notin D'\}$. The quotient $\overline{M} = M/M_0$ has the diagram D'. If $\pi : M \to \overline{M}$ is the canonical projection, the element $\pi(v_j) \in \overline{M}$ will be also denoted by $\overline{v_j}$.

As a subdiagram is determined by its set of vertices, we can define the set theoretic operations for the subdiagrams of D by the corresponding operations on their sets of vertices. The open subdiagrams define a topology on the (finite) set of subdiagrams of D, and the open and closed subdiagrams are complementary.

The modules $\operatorname{Rad} M$ and $\operatorname{top} M = M/\operatorname{Rad}(M)$ can be also easily described in terms of subdiagrams. Let D_{Rad} be the open subdiagram of D with vertices $V_{\text{Rad}} = \{j \in V \,|\, \text{ there exists an edge } e(i,j) \text{ in } D\}$, and let D_{top} be the closed subdiagram of D with vertices $V_{\text{top}} = V \backslash V_{\text{Rad}}$ and no edges. Then it is easily seen that $\operatorname{Rad} M$ has the diagram D_{Rad} with respect to the basis $\{v_j \,|\, j \in V_{\text{Rad}}\}$, and $\operatorname{top} M$ has the diagram D_{top} with respect to the basis $\{\overline{v_j} \,|\, j \in V_{\text{top}}\}$.

Finitely, we introduce the concept of cutting and pasting as in [12].

4 Projective Resolutions

For a simple R-module S_i, let $\ldots \xrightarrow{d_1^{(i)}} Q_1^{(i)} \xrightarrow{d_0^{(i)}} Q_0^{(i)} \xrightarrow{d_{-1}^{(i)}} S_i \to 0$ denote the minimal projective resolution of S_i. We will write $\Omega^n(S_i)$ for its n-th syzygy $\operatorname{Im}(d_{n-1}^{(i)}(M))$, $n \geqslant 0$. The multiplication on the right by an element $x \in e_i R e_j$ induces a homomorphism from P_i into P_j, we denote this homomorphism by the same letter x.

In this section, we use the diagrammatic method to find the minimal projective resolutions and syzygies of S_i. Since the vertices of a diagram of an R-module M correspond to a basis of M and the edges reflect the R-module structure on M, we can consider diagrams and diagram maps rather than modules and homomorphisms. Diagram homomorphisms (obvious in our context) can be formally defined as in [12, Def. 2.6].

We identify S_0 with $\langle \alpha^s \rangle \subset P_0$. Set $Q_0^{(0)} = P_0$ and define an epimorphism $d_{-1}^{(0)} : Q_0^{(0)} \to S_0$ by $d_{-1}^{(0)}(e_0) = \alpha^s$. Then we have an exact sequence

$$\begin{array}{ccccc} S_0 & & & & S_1 \\ & {\scriptstyle \alpha^{s-1}}\diagdown & & \diagup{\scriptstyle \lambda\delta b} & \\ & & S_0 & & \end{array} \hookrightarrow Q_0^{(0)} \xrightarrow{\alpha^s} S_0 \,,$$

where the open subdiagram on the left represents $\Omega^1(S_0) \subset P_0$. Since $\ker d_{-1}^{(0)} \subset \operatorname{Rad} P_0$, $d_{-1}^{(0)}$ defines a projective cover of S_0. Set $Q_1^{(0)} = P_0 \bigoplus P_1$ and define an epimorphism $d_0^{(0)} : Q_1^{(0)} \to \Omega^1(S_0)$ by $d_0^{(0)}(e_0, 0) = \alpha$ and $d_0^{(0)}(0, e_1) = \beta$. We have an exact sequence

$$\begin{array}{ccccccccc} S_1 & & & & S_0 & & & & S_1 \\ & {\scriptstyle \lambda\delta b}\diagdown & & \diagup{\scriptstyle \alpha} & & {\scriptstyle \beta}\diagdown & & \diagup{\scriptstyle \rho^{t-1}} & \\ & & S_0 & & & & S_1 & & \end{array} \hookrightarrow Q_1^{(0)} \xrightarrow{(\alpha,\beta)} \Omega^1(S_0) \,,$$

where the left diagram represents

$$\Omega^2(S_0) = \ker d_0^{(0)} = \langle (\beta, 0), (\delta\beta, 0), \ldots, ((\lambda\delta\beta)^k, 0), \\ (-\alpha^{s-1}, \lambda\delta b), (0, (\beta\lambda\delta)^k), (0, \rho^{t-1}), \ldots, (0, \rho) \rangle \subset P_0 \bigoplus P_1 \,.$$

Since $\ker d_0^{(0)} \subset \operatorname{Rad} Q_1^{(0)}$, $d_0^{(0)}$ also defines a projective cover of $\Omega^1(S_0)$. Continuing in the same manner and using the induction, the reader will prove the following propositions.

Proposition 4.1. a) *The diagrams of* $\Omega^0(S_0)$ *and* $\Omega^1(S_0)$ *are, respectively,*

$$S_0 \qquad \textit{and} \qquad \begin{array}{ccccc} S_0 & & & & S_1 \\ & {\scriptstyle \alpha^{s-1}}\diagdown & & \diagup{\scriptstyle \lambda\delta b} & \\ & & S_0 & & \end{array} \,.$$

b) *Let* $m \geqslant 2$ *be an integer. Suppose* $m \equiv r \pmod 6$ *with* $0 \leqslant r \leqslant 5$. *Let* D *be the diagram of* $\Omega^{m-2}(S_0)$. *Then the diagram of the module* $\Omega^m(S_0)$ *can be obtained from* D *by adjoining or omitting some subdiagrams (depending on* r*) on both sides of* D. *The following table shows the subdiagrams to adjoin* $(+)$ *and to omit* $(-)$ *on the left and on the right side of* D*:*

$r=0$	$\begin{array}{ccc} S_2 & & \\ & {}_{\lambda}\searrow & \\ & & S_0 \end{array}$	$-$	D	$+$	$\begin{array}{ccccc} S_2 & & & & S_0 \\ & {}_{\lambda}\searrow & & \nearrow_{\alpha^{s-1}} & \\ & & S_0 & & \end{array}$
$r=1$	$\begin{array}{ccccc} S_0 & & & & S_2 \\ & {}_{\alpha^{s-1}}\searrow & & \nearrow_{\lambda} & \\ & & S_0 & & \end{array}$	$+$	D	$+$	$\begin{array}{ccccc} S_0 & & & & S_1 \\ & {}_{\alpha}\searrow & & \nearrow_{\lambda\delta b} & \\ & & S_0 & & \end{array}$
$r=2$	$\begin{array}{ccccc} S_1 & & & & S_0 \\ & {}_{\lambda\delta b}\searrow & & \nearrow_{\alpha} & \\ & & S_0 & & \end{array}$	$+$	D	$+$	$\begin{array}{ccccc} S_0 & & & & S_1 \\ & {}_{\beta}\searrow & & \nearrow_{\rho^{t-1}} & \\ & & S_1 & & \end{array}$
$r=3$	$\begin{array}{ccccc} S_1 & & & & S_0 \\ & {}_{\rho^{t-1}}\searrow & & \nearrow_{\beta} & \\ & & S_1 & & \end{array}$	$+$	D	$+$	$\begin{array}{ccccc} S_1 & & & & S_2 \\ & {}_{\rho}\searrow & & \nearrow_{\beta\lambda d} & \\ & & S_1 & & \end{array}$
$r=4$	$\begin{array}{ccccc} S_2 & & & & S_1 \\ & {}_{\beta\lambda d}\searrow & & \nearrow_{\rho} & \\ & & S_1 & & \end{array}$	$+$	D	$+$	$\begin{array}{ccc} S_1 & & \\ & {}_{\delta}\searrow & \\ & & S_2 \end{array}$
$r=5$	$\begin{array}{ccc} & & S_1 \\ & \nearrow_{\delta} & \\ S_2 & & \end{array}$	$+$	D	$-$	$\begin{array}{ccc} & & S_2 \\ & \nearrow_{\lambda} & \\ S_0 & & \end{array}$

Proposition 4.2. a) *The diagrams of* $\Omega^0(S_1)$ *and* $\Omega^1(S_1)$ *are, respectively,*

$$S_1 \qquad \textit{and} \qquad \begin{array}{ccccc} S_2 & & & & S_1 \\ & {}_{\beta\lambda d}\searrow & & \nearrow_{\rho^{t-1}} & \\ & & S_1 & & \end{array} .$$

b) *Let* $m \geqslant 2$ *be an integer. Suppose* $m \equiv r \pmod 6$ *with* $0 \leqslant r \leqslant 5$. *Let* D *be the diagram of* $\Omega^{m-2}(S_1)$. *Then the diagram of the module* $\Omega^m(S_1)$ *can be obtained from* D *by adjoining or omitting some subdiagrams (depending on* r*) on both sides of* D. *The following table shows the subdiagrams to adjoin* $(+)$ *and to omit* $(-)$ *on the left and on the right side of* D*:*

$r=0$	$\begin{array}{ccccc} S_1 & & & & S_0 \\ & \rho^{t-1}\searrow & & \nearrow\beta & \\ & & S_1 & & \end{array} + D + \begin{array}{ccccc} S_0 & & & & S_1 \\ & \alpha\searrow & & \nearrow\lambda\delta b & \\ & & S_0 & & \end{array}$
$r=1$	$\begin{array}{ccccc} S_2 & & & & S_1 \\ & \beta\lambda d\searrow & & \nearrow\rho & \\ & & S_1 & & \end{array} + D + \begin{array}{ccccc} S_0 & & & & S_1 \\ & \beta\searrow & & \nearrow\rho^{t-1} & \\ & & S_1 & & \end{array}$
$r=2$	$\begin{array}{ccc} & & S_1 \\ & \nearrow\delta & \\ S_2 & & \end{array} + D + \begin{array}{ccccc} S_1 & & & & S_2 \\ & \rho\searrow & & \nearrow\beta\lambda d & \\ & & S_1 & & \end{array}$
$r=3$	$\begin{array}{ccc} S_2 & & \\ & \lambda\searrow & \\ & & S_0 \end{array} - D + \begin{array}{ccc} S_1 & & \\ & \delta\searrow & \\ & & S_2 \end{array}$
$r=4$	$\begin{array}{ccccc} S_0 & & & & S_2 \\ & \alpha^{s-1}\searrow & & \nearrow\lambda & \\ & & S_0 & & \end{array} + D - \begin{array}{ccc} & & S_2 \\ & \nearrow\lambda & \\ S_0 & & \end{array}$
$r=5$	$\begin{array}{ccccc} S_1 & & & & S_0 \\ & \lambda\delta b\searrow & & \nearrow\alpha & \\ & & S_0 & & \end{array} + D + \begin{array}{ccccc} S_2 & & & & S_0 \\ & \lambda\searrow & & \nearrow\alpha^{s-1} & \\ & & S_0 & & \end{array}$

For example, the diagram of $\Omega^3(S_1)$ is obtained from that of $\Omega^1(S_1)$ by omitting the edge λ on the left and adding the edge δ on the right:

$$\overbrace{\begin{array}{ccccc} S_2 & & & & S_1 \\ & b\beta\lambda\searrow & & \nearrow\rho^{t-1} & \\ & & S_1 & & \end{array}}^{\Omega^1(S_1)} \quad\mapsto\quad \overbrace{\begin{array}{ccccccc} S_0 & & & & S_1 & & \\ & b\beta\searrow & & \nearrow\rho^{t-1} & & \delta\searrow & \\ & & S_1 & & & & S_2 \end{array}}^{\Omega^3(S_1)} .$$

Let

$$B_{\bullet\bullet}^{(0)} = \{\, B_{ij},\ \Delta_{ij}^{(h)} \colon B_{ij} \to B_{i-1,j}\ ,\ \Delta_{ij}^{(v)} \colon B_{ij} \to B_{i,j-1} \,\}$$

be the bicomplex (4.1) lying in the first quadrant of the plane (i.e. $B_{ij} = 0$ if $i < 0$ or $j < 0$), where i denotes the column index and j denotes the row index. The bicomplex $B_{\bullet\bullet}^{(0)}$ is invariant with respect to translations by the vector $(5,1)$ anywhere below the main diagonal and by the vector $(1,5)$ above the main diagonal (modulo the minus signs which are added to all $\Delta_{ij}^{(h)}$ with odd j). In particular, these rules define the boundary maps of the bicomplex as follows ($i, j \geqslant 0$):

$$
\begin{array}{lcll}
\Delta^{(h)}_{5j+r,j} & = - & \Delta^{(h)}_{5(j+1)+r,j+1} & \text{for } r = 1,2,3, \\
\Delta^{(v)}_{5j+3,j+1} & = & \Delta^{(v)}_{5(j+1)+3,j+2}\,, & \\
\Delta^{(h)}_{5j+r,j+1} & = - & \Delta^{(h)}_{5(j+1)+r,j+2} & \text{for } r = 4,5; \\
\Delta^{(v)}_{i,5i+r} & = & \Delta^{(v)}_{i+1,5(i+1)+r} & \text{for } r = 1,2,3,4, \\
\Delta^{(h)}_{i+1,5i+4} & = - & \Delta^{(h)}_{i+2,5(i+1)+4}\,, & \\
\Delta^{(v)}_{i+1,5i+5} & = & \Delta^{(v)}_{i+2,5(i+1)+5}\,. &
\end{array}
$$

The bicomplex $B^{(0)}_{\bullet\bullet}$ is also invariant (again modulo the minus signs) with respect to translations by the vector $(1,1)$ strictly inside the non-zero part.

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\lambda\delta}\big\downarrow & & {\scriptstyle\rho}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow & & \\
 & & P_0 & \xleftarrow{-\beta l} & P_1 & \xleftarrow{-\rho^{t-1}} & P_1 & \xleftarrow{-\lambda\delta b} & P_0 & \xleftarrow{-\alpha^{s-1}} & P_0 & \xleftarrow{-\beta} & \cdots \\
 & & {\scriptstyle\lambda\delta}\big\downarrow & & {\scriptstyle\rho}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\lambda\delta b}\big\downarrow & & \\
P_2 & \xleftarrow{\beta\lambda d} & P_1 & \xleftarrow{\rho^{t-1}} & P_1 & \xleftarrow{\lambda\delta b} & P_0 & \xleftarrow{\alpha^{s-1}} & P_0 & \xleftarrow{\beta} & P_1 & \xleftarrow{\rho} & \cdots \\
{\scriptstyle\delta}\big\downarrow & & {\scriptstyle\rho}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\lambda\delta b}\big\downarrow & & {\scriptstyle\rho^{t-1}}\big\downarrow & & \\
P_1 & \xleftarrow{-\rho^{t-1}} & P_1 & \xleftarrow{-\lambda\delta b} & P_0 & \xleftarrow{-\alpha^{s-1}} & P_0 & \xleftarrow{-\beta} & P_1 & \xleftarrow{-\rho} & P_1 & \xleftarrow{-\lambda\delta} & \cdots \\
{\scriptstyle\rho}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\lambda\delta b}\big\downarrow & & {\scriptstyle\rho^{t-1}}\big\downarrow & & {\scriptstyle\beta l}\big\downarrow & & \\
P_1 & \xleftarrow{\lambda\delta b} & P_0 & \xleftarrow{\alpha^{s-1}} & P_0 & \xleftarrow{\beta} & P_1 & \xleftarrow{\rho} & P_1 & \xleftarrow{\lambda\delta} & P_0 & \xleftarrow{\alpha} & \cdots \\
{\scriptstyle\beta}\big\downarrow & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\lambda\delta b}\big\downarrow & & {\scriptstyle\rho^{t-1}}\big\downarrow & & {\scriptstyle\beta l}\big\downarrow & & {\scriptstyle\alpha^{s-1}}\big\downarrow & & \\
P_0 & \xleftarrow{-\alpha^{s-1}} & P_0 & \xleftarrow{-\beta} & P_1 & \xleftarrow{-\rho} & P_1 & \xleftarrow{-\lambda\delta} & P_0 & \xleftarrow{-\alpha} & P_0 & \xleftarrow{-\beta} & \cdots \\
{\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\lambda\delta b}\big\downarrow & & {\scriptstyle\rho^{t-1}}\big\downarrow & & {\scriptstyle\beta\lambda d}\big\downarrow & & & & & & \\
P_0 & \xleftarrow{\beta} & P_1 & \xleftarrow{\rho} & P_1 & \xleftarrow{\delta} & P_2 & & & & & &
\end{array}
\tag{4.1}
$$

Proposition 4.3. *The minimal projective resolution of the R-module S_0 coincides with the total complex of the bicomplex $B^{(0)}_{\bullet\bullet}$.*

To describe the minimal projective resolution of the module S_1 we consider the bicomplex $B^{(1)}_{\bullet\bullet}$ in (4.2) which has properties similar to that of the bicomplex (4.1).

$$
\begin{array}{ccccccccccccc}
 & & & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & & & {\scriptstyle\lambda\delta}\downarrow & & {\scriptstyle\rho}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow & & \\
 & & P_2 & \xleftarrow{\beta\lambda d} & P_1 & \xleftarrow{\rho^{t-1}} & P_1 & \xleftarrow{\lambda\delta b} & P_0 & \xleftarrow{\alpha^{s-1}} & P_0 & \xleftarrow{\beta l} & \cdots \\
 & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\rho}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\lambda\delta}\downarrow & & \\
 & & P_1 & \xleftarrow{-\rho^{t-1}} & P_1 & \xleftarrow{-\lambda\delta b} & P_0 & \xleftarrow{-\alpha^{s-1}} & P_0 & \xleftarrow{-\beta l} & P_0 & \xleftarrow{-\rho} & \cdots \\
 & & {\scriptstyle\rho}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\lambda\delta}\downarrow & & {\scriptstyle\rho^{t-1}}\downarrow & & \\
 & & P_1 & \xleftarrow{\lambda\delta b} & P_0 & \xleftarrow{\alpha^{s-1}} & P_0 & \xleftarrow{\beta l} & P_1 & \xleftarrow{\rho} & P_1 & \xleftarrow{\lambda\delta} & \cdots \\
 & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\lambda\delta}\downarrow & & {\scriptstyle\rho^{t-1}}\downarrow & & {\scriptstyle\beta l}\downarrow & & \\
 & & P_0 & \xleftarrow{-\alpha^{s-1}} & P_0 & \xleftarrow{-\beta l} & P_1 & \xleftarrow{-\rho} & P_1 & \xleftarrow{-\lambda\delta} & P_0 & \xleftarrow{-\alpha} & \cdots \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\lambda\delta}\downarrow & & {\scriptstyle\rho^{t-1}}\downarrow & & {\scriptstyle\beta l}\downarrow & & {\scriptstyle\alpha^{s-1}}\downarrow & & \\
 & & P_0 & \xleftarrow{\beta l} & P_1 & \xleftarrow{\rho} & P_1 & \xleftarrow{\lambda\delta} & P_0 & \xleftarrow{\alpha} & P_0 & \xleftarrow{\beta} & \cdots \\
 & & {\scriptstyle\lambda\delta}\downarrow & & {\scriptstyle\rho^{t-1}}\downarrow & & {\scriptstyle\beta l}\downarrow & & {\scriptstyle\alpha^{s-1}}\downarrow & & {\scriptstyle\lambda\delta b}\downarrow & & \\
P_2 & \xleftarrow{-\beta\lambda d} & P_1 & \xleftarrow{-\rho} & P_1 & \xleftarrow{-\lambda\delta} & P_0 & \xleftarrow{-\alpha} & P_0 & \xleftarrow{-\beta} & P_1 & \xleftarrow{-\rho} & \cdots \\
{\scriptstyle\delta}\downarrow & & {\scriptstyle\rho^{t-1}}\downarrow & & {\scriptstyle\beta\lambda d}\downarrow & & & & & & & & \\
P_1 & \xleftarrow{\rho} & P_1 & \xleftarrow{\delta} & P_2 & & & & & & & &
\end{array}
\tag{4.2}
$$

Proposition 4.4. *The minimal projective resolution of the R-module S_1 coincides with the total complex of the bicomplex $B^{(1)}_{\bullet\bullet}$.*

We emphasize that the main difficulty of this step of our method is not in proving, but in finding the bicomplex, whose periodic properties can be rather complicated. Although Propositions 4.3 and 4.4 can also be proved by a straightforward verification of exactness or by using a spectral sequence as in [5], our version of the diagrammatic method seems to be the most convenient tool to find the bicomplex.

The following proposition can be verified directly without diagrams.

Proposition 4.5. *The module S_2 is Ω-periodic with period 6 and its minimal projective resolution is*

$$\ldots \xrightarrow{\lambda} P_2 \xrightarrow{(\delta\beta\lambda)^k} P_2 \xrightarrow{\delta} P_1 \xrightarrow{\rho} P_1 \xrightarrow{\beta} P_0 \xrightarrow{\alpha} P_0 \xrightarrow{\lambda} P_2 \longrightarrow S_2 \longrightarrow 0.$$

Corollary 4.6. *Let $m \geqslant 0$ be an integer. Suppose $m = 6q + r$ with $q, r \in \mathbb{Z}$ and $0 \leqslant r \leqslant 5$. Then we have:*

a) $Q_m^{(0)} \simeq P_0^n \bigoplus P_1^{n'} \bigoplus P_2^{n''}$ *with*

$$(n, n', n'') = \begin{cases} (2q+1, 2q, 0), & \textit{if } r = 0, \\ (2q+1, 2q+1, 0), & \textit{if } r = 1, \\ (2q+1, 2q+2, 0), & \textit{if } r = 2, \\ (2q+1, 2q+2, 1), & \textit{if } r = 3 \textit{ or } 4, \\ (2q+2, 2q+2, 0), & \textit{if } r = 5; \end{cases}$$

b) $Q_m^{(1)} \simeq P_0^n \bigoplus P_1^{n'} \bigoplus P_2^{n''}$ *with*

$$(n, n', n'') = \begin{cases} (2q, 2q+1, 0), & \textit{if } r = 0, \\ (2q, 2q+1, 1), & \textit{if } r = 1 \textit{ or } 2, \\ (2q+1, 2q+1, 0), & \textit{if } r = 3, \\ (2q+2, 2q+1, 0), & \textit{if } r = 4, \\ (2q+2, 2q+2, 0), & \textit{if } r = 5. \end{cases}$$

The following corollary gives the dimensions of $\operatorname{Ext}_R^m(S_i, S_j)$.

Corollary 4.7. *Let $m \geqslant 0$ be an integer. Suppose $m = 6q + r$ with $q, r \in \mathbb{Z}$ and $0 \leqslant r \leqslant 5$. Then*

a)
$$\begin{aligned} \dim_K \operatorname{Ext}_R^m(S_0, S_0) &= \dim_K \operatorname{Ext}_R^m(S_1, S_1) \\ &= \begin{cases} 2q+1, & \textit{if } r = 0, 1, 2, 3, 4, \\ 2q+2, & \textit{if } r = 5; \end{cases} \end{aligned}$$

b)
$$\dim_K \operatorname{Ext}_R^m(S_0, S_1) = \begin{cases} 2q, & \textit{if } r = 0, \\ 2q+1, & \textit{if } r = 1, \\ 2q+2, & \textit{if } r = 2, 3, 4, 5; \end{cases}$$

c)
$$\dim_K \operatorname{Ext}_R^m(S_1, S_0) = \begin{cases} 2q, & \textit{if } r = 0, 1, 2, \\ 2q+1, & \textit{if } r = 3, \\ 2q+2, & \textit{if } r = 4, 5; \end{cases}$$

d)
$$\dim_K \operatorname{Ext}_R^m(S_2, S_2) = \begin{cases} 1, & \textit{if } r = 0, 5, \\ 0, & \textit{if } r = 1, 2, 3, 4; \end{cases}$$

e)
$$\begin{aligned} \dim_K \operatorname{Ext}_R^m(S_0, S_2) &= \dim_K \operatorname{Ext}_R^m(S_2, S_1) \\ &= \begin{cases} 1, & \textit{if } r = 3, 4, \\ 0, & \textit{if } r = 0, 1, 2, 5; \end{cases} \end{aligned}$$

f) $$\begin{aligned}\dim_K \operatorname{Ext}^m_R(S_2,S_0) &= \dim_K \operatorname{Ext}^m_R(S_1,S_2)\\ &= \begin{cases}1, & \text{if } \ r=1,2,\\ 0, & \text{if } \ r=0,3,4,5.\end{cases}\end{aligned}$$

Remark 4.8. By Proposition 4.3, we have $Q^{(0)}_m = \bigoplus_{i+j=m} B_{ij}$. The modules in this direct sum will be always ordered with respect to the first index, for example, we write $Q^{(0)}_3 = B_{03}\oplus B_{12}\oplus B_{21}\oplus B_{30} = P_1\oplus P_0\oplus P_1\oplus P_2$. We use similar notation for modules $Q^{(1)}_m$. The simple direct summands of $\operatorname{top}\Omega^m(S_i) \simeq \operatorname{top} Q^{(i)}_m$ will be ordered in the same way; for example, $\operatorname{top} Q^{(0)}_3 = S_1\oplus S_0\oplus S_1\oplus S_2$. We call such decompositions of $Q^{(i)}_m$ and $\operatorname{top} Q^{(i)}_m$ the *canonical* decompositions.

5 Generators

In this section, we indicate a finite set of generators for the Yoneda algebra:

$$\mathcal{Y}(R) = \mathcal{E}xt(R/J(R)) = \bigoplus_{m\geqslant 0}\bigoplus_{i,j=0}^{2} \operatorname{Ext}^m_R(S_i,S_j).$$

Let us recall some facts and notation related to the Yoneda algebra (see also [15, Chapter 2]). Since S_j is a simple R-module, we have $\operatorname{Ext}^m_R(S_i,S_j) \simeq \operatorname{Hom}_R(\Omega^m(S_i),S_j)$. Let ψ be an element of $\operatorname{Ext}^m_R(S_i,S_j)$. Its image $\widehat{\psi}$ in $\operatorname{Hom}_R(\Omega^m(S_i),S_j)$ induces a morphism of projective resolutions $\{f_l : Q^{(i)}_{m+l-1} \to Q^{(j)}_{l-1} \,|\, l\geqslant 1\}$ and a homomorphism $f_0 : Q^{(i)}_{m-1} \to P_j$. We have a commutative diagram:

$$\begin{array}{ccccc} Q^{(i)}_m & \xrightarrow{d^{(i)}_{m-1}} & \Omega^m(S_i) & \subset & Q^{(i)}_{m-1} \\ \downarrow f_1 & & \downarrow \widehat{\psi} & & \downarrow f_0 \\ Q^{(j)}_0 & \xrightarrow{d^{(j)}_{-1}} & S_j & \subset & P_j \end{array} \quad . \tag{5.1}$$

We see that $\widehat{\psi}$ can be represented by the outer square of (5.1) because this commutative square uniquely defines the map $\widehat{\psi}$. Moreover, $\widehat{\psi}$ is uniquely defined by providing only a homomorphism $f_1 : Q^{(i)}_m \to Q^{(j)}_0$ such that $d^{(j)}_{-1}f_1$ annihilates $\operatorname{Ker} d^{(i)}_{m-1}$. In this case we write $\widehat{\psi} = \operatorname{sq}\big(Q^{(i)}_m \xrightarrow{f_1} Q^{(j)}_0\big)$. The homomorphisms

$$\Omega^l(\widehat{\psi}) : \Omega^{m+l}(S_i) \to \Omega^l(S_j), \quad \Omega^l(\widehat{\psi}) = f_l|_{\Omega^{m+l}(S_i)},$$

are called *the* Ω*-translates of* $\widehat{\psi}$. We have $\Omega^l(\widehat{\psi}) = \mathrm{sq}\big(Q^{(i)}_{m+l} \xrightarrow{f_{l+1}} Q^{(j)}_l\big)$. If $\varphi \in \mathrm{Ext}^n_R(S_j, S_e) \simeq \mathrm{Hom}_R(\Omega^n(S_j), S_e)$, the Yoneda product $\varphi\psi \in \mathrm{Ext}^{m+n}_R(S_i, S_e)$ has the image $\widehat{\varphi\psi} = \widehat{\varphi} \cdot \Omega^n(\widehat{\psi})$ in $\mathrm{Hom}_R(\Omega^{m+n}(S_i), S_e)$. Moreover, if $\widehat{\varphi} = \mathrm{sq}\big(Q^{(j)}_n \xrightarrow{g} Q^{(e)}_0\big)$, then $\widehat{\varphi\psi} = \mathrm{sq}\big(Q^{(i)}_{m+n} \xrightarrow{gf_{n+1}} Q^{(e)}_0\big)$. Although the maps f_l and the Ω-translates are not uniquely determined by $\widehat{\psi}$, it is easily seen that the resulting map $\widehat{\varphi\psi}$ does not depend on their choice. Since R is a QF-algebra, we can also translate the maps from left to right: any map $\rho : \Omega^{m+l}(S_i) \to \Omega^l(S_j)$ induces a map $\tilde{\rho} : \Omega^m(S_i) \to S_j$ such that $\rho = \Omega^l(\tilde{\rho})$.

Consider the homogeneous elements of $\mathcal{Y}(R)$ defined as follows:

$$\begin{gathered}
x_1 \in \mathrm{Ext}^1_R(S_0,S_0),\ x_2 \in \mathrm{Ext}^1_R(S_0,S_1),\ x_3 \in \mathrm{Ext}^1_R(S_1,S_1),\\
x_4 \in \mathrm{Ext}^1_R(S_1,S_2),\ x_5 \in \mathrm{Ext}^1_R(S_2,S_0),\\
y_1 \in \mathrm{Ext}^2_R(S_0,S_0),\ y_2 \in \mathrm{Ext}^2_R(S_1,S_1),\\
z \in \mathrm{Ext}^3_R(S_1,S_0),\ w \in \mathrm{Ext}^6_R(S_2,S_2);\\
\widehat{x}_1 = \mathrm{sq}(Q^{(0)}_1 \xrightarrow{(1,0)} Q^{(0)}_0),\ \widehat{x}_2 = \mathrm{sq}(Q^{(0)}_1 \xrightarrow{(0,1)} Q^{(1)}_0),\ \widehat{x}_3 = \mathrm{sq}(Q^{(1)}_1 \xrightarrow{(0,1)} Q^{(1)}_0),\\
\widehat{x}_4 = \mathrm{sq}(Q^{(1)}_1 \xrightarrow{(1,0)} Q^{(2)}_0),\ \widehat{x}_5 = \mathrm{sq}(Q^{(2)}_1 \xrightarrow{\mathrm{id}} Q^{(0)}_0),\\
\widehat{y}_1 = \mathrm{sq}(Q^{(0)}_2 \xrightarrow{(0,-1,0)} Q^{(0)}_0), \widehat{y}_2 = \mathrm{sq}(Q^{(1)}_2 \xrightarrow{(1,0)} Q^{(1)}_0),\\
\widehat{z} = \mathrm{sq}(Q^{(1)}_3 \xrightarrow{(1,0)} Q^{(0)}_0),\ \widehat{w} = \mathrm{sq}(Q^{(2)}_6 \xrightarrow{\mathrm{id}} Q^{(2)}_0).
\end{gathered}$$

It will cause no confusion to use the same letters as for the elements of $\mathcal{E}$ (we will use only $x_i, y_i, z, w \in \mathcal{Y}$ in this section and only $x_i, y_i, z, w \in \mathcal{E}$ in the proof of Proposition 6.1). To show how we compute the Ω-translates, let us determine $\Omega^1(\widehat{x}_1)$. The map $\widehat{x}_1$ is defined by the right square in the diagram

$$\begin{array}{ccccc}
Q^{(0)}_2 & \xrightarrow{\begin{pmatrix}\beta & -\alpha^{s-1} & 0\\ 0 & \lambda\delta b & \rho\end{pmatrix}} & Q^{(0)}_1 & \xrightarrow{(\alpha,\beta)} & Q^{(0)}_0\\
U\big\downarrow ? & & (1,0)\big\downarrow & & \alpha^{s-1}\big\downarrow\\
Q^{(0)}_1 & \xrightarrow[(\alpha,\beta)]{} & Q^{(0)}_0 & \xrightarrow[\alpha^s]{} & P_0\ .
\end{array}$$

We have to find a map $U : P_1 \bigoplus P_0 \bigoplus P_1 \to P_0 \bigoplus P_1$ such that the diagram commutes, and therefore $\Omega^1(\widehat{x}_1) = \mathrm{sq}(Q^{(0)}_2 \xrightarrow{U} Q^{(0)}_1)$. Writing the corresponding matrix equation, we see that we can take, for example,

$$U = \begin{pmatrix} 0 & -\alpha^{s-2} & 0\\ 1 & 0 & 0\end{pmatrix}. \tag{5.2}$$

Proposition 5.1. *The extension groups below have the following K-bases:*

$$
\begin{array}{ll}
\mathrm{Ext}^1_R(S_0,S_0)=\langle x_1\rangle, & \mathrm{Ext}^1_R(S_0,S_1)=\langle x_2\rangle,\\
\mathrm{Ext}^1_R(S_1,S_1)=\langle x_3\rangle, & \mathrm{Ext}^1_R(S_1,S_2)=\langle x_4\rangle,\\
\mathrm{Ext}^1_R(S_2,S_0)=\langle x_5\rangle, & \mathrm{Ext}^2_R(S_0,S_0)=\langle y_1\rangle,\\
\mathrm{Ext}^2_R(S_1,S_1)=\langle y_2\rangle, & \mathrm{Ext}^2_R(S_0,S_1)=\langle x_2x_1, x_3x_2\rangle,\\
\mathrm{Ext}^2_R(S_1,S_2)=\langle x_4x_3\rangle, & \mathrm{Ext}^2_R(S_2,S_0)=\langle x_1x_5\rangle,\\
\mathrm{Ext}^3_R(S_0,S_0)=\langle x_1y_1\rangle, & \mathrm{Ext}^3_R(S_0,S_1)=\langle x_3x_2x_1, x_2y_1\rangle,\\
\mathrm{Ext}^3_R(S_0,S_2)=\langle x_4x_3x_2\rangle, & \mathrm{Ext}^3_R(S_1,S_1)=\langle x_3y_2\rangle,\\
\mathrm{Ext}^3_R(S_1,S_0)=\langle z\rangle, & \mathrm{Ext}^3_R(S_2,S_1)=\langle x_2x_1x_5\rangle,\\
\mathrm{Ext}^4_R(S_0,S_0)=\langle y_1^2\rangle, & \mathrm{Ext}^4_R(S_0,S_1)=\langle y_2x_2x_1, x_3x_2y_1\rangle,\\
\mathrm{Ext}^4_R(S_0,S_2)=\langle x_4x_3x_2x_1\rangle, & \mathrm{Ext}^4_R(S_1,S_1)=\langle y_2^2\rangle,\\
\mathrm{Ext}^4_R(S_1,S_0)=\langle zx_3, x_1z\rangle, & \mathrm{Ext}^4_R(S_2,S_1)=\langle x_3x_2x_1x_5\rangle,\\
\mathrm{Ext}^5_R(S_0,S_0)=\langle x_1y_1^2, zx_3x_2\rangle, & \mathrm{Ext}^5_R(S_0,S_1)=\langle x_3y_2x_2x_1, x_2y_1^2\rangle,\\
\mathrm{Ext}^5_R(S_1,S_0)=\langle x_1zx_3, y_1z\rangle, & \mathrm{Ext}^5_R(S_1,S_1)=\langle x_3y_2^2, x_2x_1z\rangle,
\end{array}
$$
$$\mathrm{Ext}^5_R(S_2,S_2)=\langle x_4x_3x_2x_1x_5\rangle.$$

Proof. We prove only that $\mathrm{Ext}^2_R(S_0,S_1)=\langle x_2x_1, x_3x_2\rangle$. The other groups are considered similarly, and we leave it to the reader.

Since $\widehat{x}_2=\mathrm{sq}(Q_1^{(0)}\xrightarrow{(0,1)}Q_0^{(1)})$ and $\Omega^1(\widehat{x}_1)=\mathrm{sq}(Q_2^{(0)}\xrightarrow{U}Q_1^{(0)})$ where U is defined in (5.2), we have $\widehat{x_2x_1}=\widehat{x}_2\cdot\Omega^1(\widehat{x}_1)=\mathrm{sq}(Q_2^{(0)}\xrightarrow{(0,1)U}Q_0^{(1)})=\mathrm{sq}(Q_2^{(0)}\xrightarrow{(1,0,0)}Q_0^{(1)})$. In the same manner we obtain $\widehat{x_3x_2}=\widehat{x}_3\cdot\Omega^1(\widehat{x}_2)=\mathrm{sq}(Q_2^{(0)}\xrightarrow{(0,0,1)}Q_0^{(1)})$. We see now that x_2x_1 and x_3x_2 are linearly independent. It remains to note that $\dim_K\mathrm{Ext}^2_R(S_0,S_1)=2$ by Corollary 4.7 b). □

Lemma 5.2. *Let S be a simple R-module, M_1 and M_2 be two R-modules with diagrams D_1 and D_2 with respect to the bases $\{v_{1p}\}_p\subset M_1$ and $\{v_{2q}\}_q\subset M_2$ respectively. Suppose that D' is a closed subdiagram of D_1 and an open subdiagram of D_2. Let c be a common vertex of D'_{top} and $(D_2)_{top}$. Suppose that $f:M_1\to S$ is an R-homomorphism such that the $f(v_{1j})=0$ for any vertex $j\in(D_1)_{top}$, $j\neq c$. Then there exist R-homomorphisms $g\colon M_1\to M_2$ and $f':M_2\to S$ such that $f=f'g$.*

Proof. Let M' be the submodule of M_2 defined by D', we identify it with the corresponding to D' quotient of M_1. Let $\pi:M_1\to M'$ and $i:M'\to M_2$ be the canonical epimorphism and monomorphism respectively. Set $g=i\pi$. We have $g(v_{1c})=i(\pi(v_{1c}))=i(v_{2c})=v_{2c}$ and $c\in(D_1)_{top}$ since $D'_{top}\subset(D_1)_{top}$.

As S is a simple R-module, for any $h\in\mathrm{Hom}_R(M_i,S)$ we have $h(\mathrm{Rad}\,M_i)=0$, hence $\mathrm{Hom}_R(M_i,S)\simeq\mathrm{Hom}_R(\mathrm{top}\,M_i,S)$. Denote by $\overline{h}\in\mathrm{Hom}_R(\mathrm{top}\,M_i,S)$ the image of h by this isomorphism. Since $\mathrm{top}\,M_2=\bigoplus_{j\in(D_2)_{\mathrm{top}}}K\overline{v_{2j}}$, we can define a homomorphism $\overline{f'}:\mathrm{top}\,M_2\to S$ by $\overline{f'}(\overline{v_{2c}})=f(v_{1c})$ and

$\overline{f'}(\overline{v_{2j}}) = 0$ for any $j \in (D_2)_{top}$, $j \neq c$. It is easily seen that the corresponding $f' \in \mathrm{Hom}_R(M_2, S)$ satisfies $f = f'g$. $\square$

Proposition 5.3. *The set $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, z, w\}$ generates the Yoneda algebra $\mathcal{Y}(R)$ as a K-algebra.*

Proof. We prove by induction on m that the groups $\mathrm{Ext}_R^m(S_i, S_j)$ are generated by some products of elements of $\mathcal{X}$. For $m \leqslant 5$, this follows directly from Proposition 5.1 and Corollary 4.7. Assume that $m \geqslant 6$ and that our statement holds for all $\mathrm{Ext}_R^{m'}(S_i, S_j)$ with $m' < m$, we will prove it for m.

If $j = 2$, we notice that, by Proposition 4.5, the multiplication on the left by w induces the isomorphism

$$\mathrm{Ext}_R^{m-6}(S_i, S_2) \xrightarrow{\simeq} \mathrm{Ext}_R^m(S_i, S_2),$$

and our statement for $\mathrm{Ext}_R^m(S_i, S_2)$ follows from the induction hypothesis. Similar arguments apply to the case $i = 2$. It remains to prove our statement for $i, j \in \{0, 1\}$.

Using the isomorphism $\mathrm{Ext}_R^m(S_i, S_j) \simeq \mathrm{Hom}_R(\Omega^m(S_i), S_j)$, we represent an element of the group $\mathrm{Ext}_R^m(S_i, S_j)$ by the corresponding map $f\colon \Omega^m(S_i) \to S_j$. Since $\mathrm{Hom}_R(\Omega^m(S_i), S_j) \simeq \mathrm{Hom}_R(\mathrm{top}\,(\Omega^m(S_i)), S_j)$ and $\mathrm{top}\,(\Omega^m(S_i))$ is a direct sum of simple modules, we can assume without loss of generality that f induces a non-zero map on at most one simple direct summand in the canonical decomposition of $\mathrm{top}\,\Omega^m(S_i)$ (see Remark 4.8).

First we consider the case $i = 0$.

Case 1: $m \equiv 1 \,(\mathrm{mod}\, 6)$. a) Assume that $f\colon \Omega^m(S_0) \to S_j$ induces zero maps on the extreme (the left and the right) simple direct summands of the module $\mathrm{top}\,\Omega^m(S_0)$. It follows from Proposition 4.1 that the diagram of $\Omega^{m-2}(S_0)$ is a closed subdiagram in that of $\Omega^m(S_0)$, hence $\Omega^{m-2}(S_0)$ is a quotient of $\Omega^m(S_0)$. Applying Lemma 5.2 with $M_1 = \Omega^m(S_0)$ and $M_2 = M' = \Omega^{m-2}(S_0)$, we have $f = f'g$ for some $g \in \mathrm{Hom}_R(\Omega^m(S_0), \Omega^{m-2}(S_0))$ and $f' \in \mathrm{Hom}_R(\Omega^{m-2}(S_0), S_j)$. Since $g = \Omega^{m-2}(\tilde{g})$ for some homomorphism $\tilde{g} : \Omega^2(S_0) \to S_0$, the desired statement follows from $f = f' \cdot \Omega^{m-2}(\tilde{g})$ and the induction hypothesis for $f' \in \mathrm{Hom}_R(\Omega^{m-2}(S_0), S_j) \simeq \mathrm{Ext}_R^{m-2}(S_0, S_j)$ and $\tilde{g} \in \mathrm{Hom}_R(\Omega^2(S_0), S_0) \simeq \mathrm{Ext}_R^2(S_i, S_i)$.

b) Assume now that f induces a non-zero map on the extreme left direct summand of $\mathrm{top}\,\Omega^m(S_0)$ (hence we have $j = 0$). By Proposition 4.1, the diagram of $\Omega^m(S_0)$ contains on the left side the closed subdiagram

$$D'\colon \begin{array}{ccccc} S_0 & & & & S_1 \\ & {\scriptstyle \alpha^{s-1}}\searrow & & \swarrow{\scriptstyle \lambda\delta} & \\ & & S_0 & & \end{array} \tag{5.3}$$

(the edge δ being added while constructing $\Omega^{m-2}(S_0)$ from $\Omega^{m-4}(S_0)$). Let D_1 be the diagram of $M_1 = \Omega^m(S_0)$ and let D_2 be the diagram of $M_2 = \Omega^1(S_0)$. Since the diagram D' in (5.3) is an open subdiagram of D_2, we can apply Lemma 5.2. Hence we have $f = f'\Omega^1(\tilde{g})$ for some homomorphisms $f' \in \mathrm{Hom}_R(\Omega^1(S_0), S_0)$ and $\tilde{g} : \Omega^{m-1}(S_0) \to S_0$. Now our statement follows from the induction hypothesis for $f' \in \mathrm{Hom}_R(\Omega^1(S_0), S_0) \simeq \mathrm{Ext}^1_R(S_0, S_0)$ and $\tilde{g} \in \mathrm{Hom}_R(\Omega^{m-1}(S_0), S_0) \simeq \mathrm{Ext}^{m-1}_R(S_0, S_0)$.

c) Assume that f induces a non-zero map on the extreme right direct summand of $\mathrm{top}\,\Omega^m(S_0)$ (we have now $j = 1$). In this case we take the diagrams of $\Omega^m(S_0)$ and $\Omega^1(S_0)$ as D_1 and D_2 respectively, and we define D' as follows:

$$D' : \quad \begin{array}{ccccc} S_0 & & & & S_1 \\ & \overset{\alpha}{\searrow} & & \swarrow_{\lambda\delta b} & \\ & & S_0 & & \end{array} \quad . \tag{5.4}$$

Since D' is an open subdiagram of D_2 and a closed subdiagram of D_1, we deduce from Lemma 5.2 that we have again $f = f' \cdot \Omega^1(\tilde{g})$ for some $f' \in \mathrm{Hom}_R(\Omega^1(S_0), S_1)$ and $\tilde{g} \in \mathrm{Hom}_R(\Omega^{m-1}(S_0), S_0)$. The desired statement now follows as above.

Case 2: $m \equiv 2 \,(\mathrm{mod}\, 6)$. a) In this case the diagram of $\Omega^{m-2}(S_0)$ is a closed subdiagram of the diagram of $\Omega^m(S_0)$ (see Proposition 4.1). If $f : \Omega^m(S_0) \to S_j$ induces zero maps on the extreme simple direct summands of the module $\mathrm{top}\,\Omega^m(S_0)$, we can proceed as in the case 1a).

b) By Proposition 4.1, we adjoin on the left side of diagram of $\Omega^{m-2}(S_0)$ the reversed subdiagram D' in (5.4). Therefore, if f induces a non-zero map on the extreme left direct summand of $\mathrm{top}\,\Omega^m(S_0)$, we can apply the argument of the case 1c).

c) Assume that f induces a non-zero map on the extreme right direct summand of $\mathrm{top}\,\Omega^m(S_0)$ (we have now $j = 1$). If we take the diagrams of $\Omega^m(S_0)$ and $\Omega^1(S_1)$ as D_1 and D_2 respectively and define D' as follows:

$$D' : \quad \begin{array}{ccccc} S_0 & & & & S_1 \\ & \overset{\beta}{\searrow} & & \swarrow_{\rho^{t-1}} & \\ & & S_1 & & \end{array} \quad ,$$

we deduce from Lemma 5.2 that $f = f' \cdot \Omega^1(\tilde{\rho})$ for some $f' \in \mathrm{Hom}_R(\Omega^1(S_1), S_1)$ and $\tilde{\rho} \in \mathrm{Hom}_R(\Omega^{m-1}(S_0), S_1)$. The desired statement follows.

Case 3: $m \equiv 3 \,(\mathrm{mod}\, 6)$. It is clear that the map $f : \Omega^m(S_0) \to S_j$, $j = 0, 1$, induces a zero map on the extreme right simple direct summand of the module $\mathrm{top}\,\Omega^m(S_0)$ (see Proposition 4.1). If f induces a zero map also on the extreme left simple direct summand of $\mathrm{top}\,\Omega^m(S_0)$, we can argue as in the case 1a) because the diagram of $\Omega^{m-2}(S_0)$ is a closed subdiagram of the diagram of $\Omega^m(S_0)$. If this induced map is non-zero, then we can apply the argument of the case 2c).

Case 4: $m \equiv 4 \pmod 6$. By Proposition 4.1, the canonical decomposition of the module $\operatorname{top}\Omega^m(S_0)$ has one additional summand (on the left side) with respect to the module $\operatorname{top}\Omega^{m-2}(S_0)$, namely, S_2. As the map $f\colon \Omega^m(S_0) \to S_j$, $j = 0, 1$, induces a zero map on this additional summand, we can argue as in the case 1a).

Case 5: $m \equiv 5 \pmod 6$. a) Let D be the diagram of $\Omega^{m-4}(S_0)$. By Proposition 4.1, the diagram of $\Omega^m(S_0)$ contains D as a closed subdiagram and can be obtained as follows:

$$\begin{array}{ccccccc} & & S_1 & & & & S_0 \\ & \nearrow_{\delta} & & {}_{\rho^{t-1}}\searrow & & \nearrow_{\beta} & \\ S_2 & & & & S_1 & & \end{array} \; + \; D \; + \; \begin{array}{ccccc} S_1 & & & & S_0 \\ & {}_{\rho}\searrow & & \nearrow_{\beta l} & \\ & & S_1 & & \end{array} \; . \tag{5.5}$$

If $f\colon \Omega^m(S_0) \to S_j$ induces zero maps on the extreme simple direct summands of the module $\operatorname{top}\Omega^m(S_0)$, we take $M_1 = \Omega^m(S_0)$, $M_2 = M' = \Omega^{m-4}(S_0)$ and proceed as in the case 1a).

b) Assume that f induces a non-zero map on an extreme left direct summand of $\operatorname{top}\Omega^m(S_0)$ (hence we have $j = 1$). From Proposition 4.2 it follows that the diagram

$$D'\colon \quad \begin{array}{ccccccc} & & S_1 & & & & S_0 \\ & \nearrow_{\delta} & & {}_{\rho^{t-1}}\searrow & & \nearrow_{\beta} & \\ S_2 & & & & S_1 & & \end{array}$$

is an open subdiagram of the diagram of $\Omega^3(S_1)$. Consequently, we can apply Lemma 5.2 taking $M_1 = \Omega^m(S_0)$ and $M_2 = \Omega^3(S_1)$. Then we proceed as in the case 1b).

c) If f induces a non-zero map on an extreme right direct summand of $\operatorname{top}\Omega^m(S_0)$ (hence $j = 0$), again by Proposition 4.2, the diagram

$$D'\colon \quad \begin{array}{ccccc} S_1 & & & & S_0 \\ & {}_{\rho}\searrow & & \nearrow_{\beta l} & \\ & & S_1 & & \end{array}$$

is an open subdiagram of the diagram of $\Omega^1(S_1)$. We can apply Lemma 5.2 taking $M_1 = \Omega^m(S_0)$ and $M_2 = \Omega^1(S_1)$ and proceed as above.

Case 6: $m \equiv 0 \pmod 6$. Let D be the diagram of $\Omega^{m-4}(S_0)$. By Proposition 4.1, the diagram of $\Omega^m(S_0)$ contains D as a closed subdiagram and can be obtained as follows:

$$\begin{array}{ccccc} S_0 & & & & S_1 \\ & {}_{\beta l}\diagdown & & \diagup_{\rho} & \\ & & S_1 & & \end{array} \; + \; D \; + \; \begin{array}{ccccc} S_1 & & & & S_0 \\ & {}_{\lambda\delta}\diagdown & & \diagup_{\alpha^{s-1}} & \\ & & S_0 & & \end{array} . \tag{5.6}$$

If $f\colon \Omega^m(S_0) \to S_j$ induces zero maps on the extreme simple direct summands of the module $\operatorname{top}\Omega^m(S_0)$, we argue as in the case 5a). If f induces a non-zero map on the extreme left direct summand of $\operatorname{top}\Omega^m(S_0)$ (hence $j = 0$), we can apply the argument of the case 5c), because the diagram of $\Omega^m(S_0)$ contains on the left side the same (but reversed) subdiagram as the right-side subdiagram in the case 5 (cf. (5.5)). At last, assume that f induces a non-zero map on the extreme right direct summand of $\operatorname{top}\Omega^m(S_0)$ (again we have $j = 0$). We apply Lemma 5.2 with $M_1 = \Omega^m(S_0)$, $M_2 = \Omega^1(S_0)$ and define D' as follows (cf. (5.6)):

$$\begin{array}{ccccc} S_1 & & & & S_0 \\ & {}_{\lambda\delta}\diagdown & & \diagup_{\alpha^{s-1}} & \\ & & S_0 & & \end{array} .$$

Then we complete the argument as above using the induction hypothesis.

If $i = 1$, we can prove the desired statement in the same manner as in the case $i = 0$. The details of this proof are left to the reader. □

Proposition 5.4. *The elements of $\mathcal{X} \subset \mathcal{Y}(R)$ satisfy the relations* (2.2).

Proof. We prove only $x_1^2 = \delta(s,2)y_1$. The verification of the other relations is similar and is left to the reader.

Since $\widehat{x}_1 = \operatorname{sq}(Q_1^{(0)} \xrightarrow{(1,0)} Q_0^{(0)})$ and $\Omega^1(\widehat{x}_1) = \operatorname{sq}(Q_2^{(0)} \xrightarrow{U} Q_1^{(0)})$ with U in (5.2), we have $\widehat{(x_1^2)} = \widehat{x}_1 \cdot \Omega^1(\widehat{x}_1) = \operatorname{sq}(Q_2^{(0)} \xrightarrow{(1,0)U} Q_0^{(1)}) = \operatorname{sq}(Q_2^{(0)} \xrightarrow{(0,-\alpha^{s-2},0)} Q_0^{(0)})$. If $s > 2$, this map obviously induces a zero map $\Omega^2(S_0) \to S_0$, which implies $x_1^2 = 0$. If $s = 2$, $\widehat{(x_1^2)}$ coincides with $\widehat{y}_1$, therefore $x_1^2 = y_1$. □

6 Proof of Theorem 2.1

Let $\mathcal{E} = \bigoplus_{m\geqslant 0} \mathcal{E}^m$ and $\mathcal{Y} = \bigoplus_{m\geqslant 0} \mathcal{Y}^m$ be the decompositions of $\mathcal{E}$ and $\mathcal{Y}$ into homogeneous direct summands. Let ε_i denote the idempotents of $K[\mathcal{R}]$ corresponding to the vertices $i = 0, 1, 2$ of $\mathcal{R}$ as well as their images in $\mathcal{E}$. We use the same notation for the idempotents $\varepsilon_i = \mathrm{id}_{S_i} \in \mathcal{Y}$.

By Propositions 5.3 and 5.4, there exists an epimorphism of graded K-algebras $\varphi : \mathcal{E} \longrightarrow \mathcal{Y}$ with $\varphi(\varepsilon_i) = \varepsilon_i$, $\varphi(x_i) = x_i$, $\varphi(y_i) = y_i$, $\varphi(z) = z, \varphi(w) = w$. To prove Theorem 2.1 it remains to show that φ is a monomorphism. It follows from the following result.

Proposition 6.1. *For any $i, j \in \{0, 1, 2\}$ and $m \geqslant 0$, we have*

$$\dim_K(\varepsilon_i \mathcal{E}^m \varepsilon_j) = \dim_K \operatorname{Ext}_R^m(S_j, S_i). \tag{6.1}$$

Proof. For $m \leqslant 5$, the relations (6.1) are verified directly. Let us assume that $m > 5$. We suppose additionally that $k > 1$, $s > 2$ and $t > 2$. The remaining cases are proved in a similar way, and we leave their proof to the reader.

a) First we consider the case $i = j = 0$. It follows from the relations (2.2) that the K-algebra $\varepsilon_0 \mathcal{E} \varepsilon_0$ is generated by the elements

$$x = x_1,\ y = y_1,\ u = zx_3x_2$$

satisfying the relations

$$xy = yx,\ yu = uy,\ x^2 = 0,\ u^2 = 0. \tag{6.2}$$

Therefore, any non-zero monomial in $\varepsilon_0 \mathcal{E}^m \varepsilon_0$ (i.e. the image of a path in $K[\mathcal{R}]$) is equal to one of the following:

$$y^\eta (xu)^\tau,\ \ y^\eta (ux)^\tau,\ \ y^\eta x(ux)^\tau,\ \ y^\eta u(xu)^\tau, \tag{6.3}$$

with $\eta, \tau \geqslant 0$.

Put $d_m = \dim_K \varepsilon_0 \mathcal{E}^m \varepsilon_0$. We claim that $d_m - d_{m-2}$ is equal to the number of monomials in (6.3) for which $\eta = 0$. Indeed, the monomials of degree m in (6.3) form a K-basis of $\varepsilon_0 \mathcal{E}^m \varepsilon_0$. Replacing η by $\eta + 1$ in the elements of the similar basis of $\varepsilon_0 \mathcal{E}^{m-2} \varepsilon_0$ gives those basis elements of $\varepsilon_0 \mathcal{E}^m \varepsilon_0$ for which $\eta > 0$. It shows that the basis elements of $\varepsilon_0 \mathcal{E}^m \varepsilon_0$ for which $\eta > 0$, are in one-to-one correspondence with the basis elements of $\varepsilon_0 \mathcal{E}^{m-2} \varepsilon_0$, which implies our claim. If $\eta = 0$, then we have for the monomials in (6.3)

$$m = 6\tau, \text{ or } \ m = 6\tau + 1, \text{ or } \ m = 6\tau + 5,$$

whence we obtain that

$$d_m - d_{m-2} = \begin{cases} 2, & \text{if } m \equiv 0 \,(\mathrm{mod}\, 6), \\ 1, & \text{if } m \equiv 1 \text{ or } 5 \,(\mathrm{mod}\, 6), \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 4.7 a) implies that the sequence $\{\dim_K \operatorname{Ext}_R^m(S_0, S_0)\}$ satisfies a similar recursive relation. The assertion (6.1) can be now established by induction on m.

b) Assume now that $i = j = 1$. Relations (2.2) imply again that the K-algebra $\varepsilon_1\mathcal{E}\varepsilon_1$ is generated by the elements

$$x = x_3, y = y_2, v = x_2x_1z$$

satisfying relations similar to (6.2) (with replacing u by v). Consequently, $\varepsilon_1\mathcal{E}\varepsilon_1 \simeq \varepsilon_0\mathcal{E}\varepsilon_0$ as graded K-algebras and our statement follows from the previous part of the proof.

c) Assume that $i = 1, j = 0$. It is clear that any non-zero monomial in $\varepsilon_1\mathcal{E}^m\varepsilon_0$ is equal to $f \cdot x_2 \cdot g$ for some monomials $f \in \varepsilon_1\mathcal{E}\varepsilon_1$ and $g \in \varepsilon_0\mathcal{E}\varepsilon_0$. Similarly to the part a) we obtain that f is equal to one of the following monomials (cf. (6.3))

$$y_2^\zeta(x_3v)^\theta,\ \ y_2^\zeta(vx_3)^\theta,\ \ y_2^\zeta x_3(vx_3)^\theta,\ \ y_2^\zeta v(x_3v)^\theta, \tag{6.4}$$

where $v = x_2x_1z$ and $\zeta, \theta \geqslant 0$. As we have the relations $y_2x_3 = x_3y_2$, $y_2v = vy_2$ and $y_2x_2 = x_2y_1$, we can assume in (6.4) that $\zeta = 0$. Moreover, $(vx_3)x_2 = x_2(x_1u)$, and hence we can additionally assume in (6.4) that $\theta = 0$. Using $x_2u = 0$, we see that any monomial in $\varepsilon_1\mathcal{E}^m\varepsilon_0$ is equal to one of the following:

$$x_2 \cdot y_1^\eta(x_1u)^\tau,\ \ x_2 \cdot y_1^\eta x_1(ux_1)^\tau,\ \ x_3 \cdot x_2 \cdot y_1^\eta(x_1u)^\tau,\ \ x_3 \cdot x_2 \cdot y_1^\eta x_1(ux_1)^\tau \tag{6.5}$$

with $\eta, \tau \geqslant 0$. Put $d_m = \dim_K \varepsilon_1\mathcal{E}^m\varepsilon_0$. As in the part a) of the proof, we see that $d_m - d_{m-2}$ is equal to the number of monomials in (6.5) for which $\eta = 0$, whence we obtain that

$$d_m - d_{m-2} = \begin{cases} 2, & \text{if } m \equiv 2 \,(\mathrm{mod}\, 6), \\ 1, & \text{if } m \equiv 1 \text{ or } 3 \,(\mathrm{mod}\, 6), \\ 0, & \text{otherwise.} \end{cases}$$

By Corollary 4.7 b), the sequence $\{\dim_K \operatorname{Ext}_R^m(S_0, S_1)\}$ satisfies a similar recursive relation, and we deduce (6.1) by induction.

d) Assume that $i = 0, j = 1$. Any non-zero monomial in $\varepsilon_0\mathcal{E}^m\varepsilon_1$ is equal to $f \cdot z \cdot g$ for some monomials $f \in \varepsilon_0\mathcal{E}\varepsilon_0$ and $g \in \varepsilon_1\mathcal{E}\varepsilon_1$; furthermore, g is equal to one of the monomials in (6.4). Since $zy_2 = y_1z$, $z(x_3v) = (ux_1)z$ and $zv = 0$, we can assume that $g = \varepsilon_1$ or $g = x_3$. Using $uz = 0$, we see that any monomial in $\varepsilon_0\mathcal{E}^m\varepsilon_1$ is equal to one of the following:

$$y_1^\eta(ux_1)^\tau \cdot z,\ \ y_1^\eta x_1(ux_1)^\tau \cdot z,\ \ y_1^\eta(ux_1)^\tau \cdot z \cdot x_3,\ \ y_1^\eta x_1(ux_1)^\tau \cdot z \cdot x_3 \tag{6.6}$$

with $\eta, \tau \geqslant 0$. Put $d_m = \dim_K \varepsilon_0 \mathcal{E}^m \varepsilon_1$. As above we see that $d_m - d_{m-2}$ is equal to the number of monomials in (6.6) for which $\eta = 0$, whence we obtain

$$d_m - d_{m-2} = \begin{cases} 2, & \text{if } m \equiv 4 \,(\mathrm{mod}\, 6), \\ 1, & \text{if } m \equiv 3 \text{ or } 5 \,(\mathrm{mod}\, 6), \\ 0, & \text{otherwise.} \end{cases}$$

It remains to observe that the sequence $\{\dim_K \mathrm{Ext}^m_R(S_1, S_0)\}$ satisfies a similar recursive relation by Corollary 4.7 c).

e) Assume now that $i = j = 2$. Relations (2.2) imply that the K-algebra $\varepsilon_2 \mathcal{E} \varepsilon_2$ is generated by the elements w and $w' = x_4 x_3 x_2 x_1 x_5$ satisfying the relations $ww' = w'w$, $(w')^2 = 0$. It is easily seen that

$$\dim_K \varepsilon_2 \mathcal{E}^m \varepsilon_2 = \begin{cases} 1, & \text{if } \ m \equiv 0 \text{ or } 5 \,(\mathrm{mod}\, 6), \\ 0, & \text{otherwise,} \end{cases}$$

and we obtain the formula (6.1) from Corollary 4.7 d).

f) Assume that $i = 2$, $j = 1$. Any non-zero monomial in $\varepsilon_2 \mathcal{E}^m \varepsilon_1$ is equal to $f \cdot x_4 \cdot g$ for some monomials $f \in \varepsilon_2 \mathcal{E} \varepsilon_2$ and $g \in \varepsilon_1 \mathcal{E} \varepsilon_1$. Since we have $w \cdot x_4 = -x_4 \cdot x_3 v$ (recall that $v = x_2 x_1 z$), $w' \cdot x_4 = 0$, $x_4 \cdot v x_3 = 0$, we see that any monomial in $\varepsilon_2 \mathcal{E}^m \varepsilon_1$ is equal to one of the following:

$$x_4 \cdot (x_3 v)^\theta, \ \ x_4 \cdot (x_3 v)^\theta x_3$$

with $\theta > 0$, whence we obtain

$$\dim_K \varepsilon_2 \mathcal{E}^m \varepsilon_1 = \begin{cases} 1, & \text{if } \ m \equiv 1 \text{ or } 2 \,(\mathrm{mod}\, 6), \\ 0, & \text{otherwise.} \end{cases}$$

By Corollary 4.7 f), we obtain (6.1).

g) At last, assume that $i = 2$, $j = 0$. As above, we can prove that any monomial in $\varepsilon_0 \mathcal{E}^m \varepsilon_2$ is equal to one of the following:

$$(u x_1)^\tau \cdot x_5, \ \ x_1 (u x_1)^\tau \cdot x_5$$

with $\tau > 0$. From this it follows that

$$\dim_K \varepsilon_0 \mathcal{E}^m \varepsilon_2 = \begin{cases} 1, & \text{if } \ m \equiv 1 \text{ or } 2 \,(\mathrm{mod}\, 6), \\ 0, & \text{otherwise.} \end{cases}$$

Using Corollary 4.7 f), we establish the desired statement. □

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7 Appendix (for the referee only). The Ω-translates of generators

7.1 The differentials in the resolutions

$$d_0^{(0)} = \begin{pmatrix} \alpha & \beta \end{pmatrix};\ d_1^{(0)} = \begin{pmatrix} \beta & -\alpha^{s-1} & 0 \\ 0 & \lambda\delta b & \rho \end{pmatrix};\ d_2^{(0)} = \begin{pmatrix} \rho & \lambda\delta b & 0 & 0 \\ 0 & \alpha & -\beta & 0 \\ 0 & 0 & \rho^{t-1} & \delta \end{pmatrix};$$

$$d_3^{(0)} = \begin{pmatrix} \delta & -\rho^{t-1} & 0 & 0 \\ 0 & \beta & \alpha^{s-1} & 0 \\ 0 & 0 & \lambda\delta b & -\rho \\ 0 & 0 & 0 & \beta\lambda d \end{pmatrix};\ d_4^{(0)} = \begin{pmatrix} \beta\lambda d & 0 & 0 & 0 \\ \rho & -\lambda\delta b & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & \rho^{t-1} & -\lambda\delta \end{pmatrix};$$

$$d_5^{(0)} = \begin{pmatrix} \lambda\delta & \rho^{t-1} & 0 & 0 & 0 \\ 0 & \beta & -\alpha^{s-1} & 0 & 0 \\ 0 & 0 & \lambda\delta b & \rho & 0 \\ 0 & 0 & 0 & \beta l & -\alpha \end{pmatrix};\ d_6^{(0)} = \begin{pmatrix} \alpha & -\beta l & 0 & 0 & 0 \\ 0 & \rho & \lambda\delta b & 0 & 0 \\ 0 & 0 & \alpha & -\beta & 0 \\ 0 & 0 & 0 & \rho^{t-1} & \lambda\delta \\ 0 & 0 & 0 & 0 & \alpha^{s-1} \end{pmatrix}.$$

$$d_0^{(1)} = \begin{pmatrix} \delta & \rho \end{pmatrix};\ d_1^{(1)} = \begin{pmatrix} -\beta\lambda d & 0 \\ \rho^{t-1} & \delta \end{pmatrix};\ d_2^{(1)} = \begin{pmatrix} \lambda\delta & -\rho \\ 0 & \beta\lambda d \end{pmatrix};$$

$$d_3^{(1)} = \begin{pmatrix} \alpha & \beta l & 0 \\ 0 & \rho^{t-1} & -\lambda\delta \end{pmatrix};\ d_4^{(1)} = \begin{pmatrix} \beta & -\alpha^{s-1} & 0 & 0 \\ 0 & \lambda\delta & \rho & 0 \\ 0 & 0 & \beta l & -\alpha \end{pmatrix};$$

$$d_5^{(1)} = \begin{pmatrix} \rho & \lambda\delta b & 0 & 0 & 0 \\ 0 & \alpha & -\beta l & 0 & 0 \\ 0 & 0 & \rho^{t-1} & \lambda\delta & 0 \\ 0 & 0 & 0 & \alpha^{s-1} & -\beta \end{pmatrix};d_6^{(1)} = \begin{pmatrix} \delta & -\rho^{t-1} & 0 & 0 & 0 & 0 \\ 0 & \beta & \alpha^{s-1} & 0 & 0 & 0 \\ 0 & 0 & \lambda\delta & -\rho & 0 & 0 \\ 0 & 0 & 0 & \beta l & \alpha & 0 \\ 0 & 0 & 0 & 0 & \lambda\delta b & -\rho \end{pmatrix}.$$

7.2 The Ω-translates of $\widehat{x}_1$

$$
\begin{array}{ccccccccc}
\xrightarrow{\begin{pmatrix}\beta\lambda d&0&0&0\\ \rho&-\lambda\delta b&0&0\\ 0&\alpha&\beta&0\\ 0&0&\rho^{t-1}&-\lambda\delta\end{pmatrix}} & Q_4^{(0)} & \xrightarrow{\begin{pmatrix}\delta&-\rho^{t-1}&0&0\\ 0&\beta&\alpha^{s-1}&0\\ 0&0&\lambda\delta b&-\rho\\ 0&0&0&\beta\lambda d\end{pmatrix}} & Q_3^{(0)} & \xrightarrow{\begin{pmatrix}\rho&\lambda\delta b&0&0\\ 0&\alpha&-\beta&0\\ 0&0&\rho^{t-1}&\delta\end{pmatrix}} & Q_2^{(0)} & \xrightarrow{\begin{pmatrix}\beta&-\alpha^{s-1}&0\\ 0&\lambda\delta b&\rho\end{pmatrix}} & Q_1^{(0)} \\
& \begin{pmatrix}0&0&0&-\delta(s,2)\\ 0&0&\alpha^{s-2}&0\\ 0&-1&0&0\\ 1&0&0&0\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0&0&\delta(s,2)&0\\ 0&1&0&0\\ 1&0&0&0\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0&-\alpha^{s-2}&0\\ 1&0&0\end{pmatrix}\Big\downarrow & & (1,0)\Big\downarrow \\
\xrightarrow[d_3^{(0)}]{} & Q_3^{(0)} & \xrightarrow[d_2^{(0)}]{} & Q_2^{(0)} & \xrightarrow[d_1^{(0)}]{} & Q_1^{(0)} & \xrightarrow[d_0^{(0)}]{} & Q_0^{(0)}
\end{array}
$$

$$
\begin{array}{ccccc}
Q_7^{(0)} & \xrightarrow{\begin{pmatrix}\alpha&-\beta l&0&0&0\\ 0&\rho&\lambda\delta b&0&0\\ 0&0&\alpha&-\beta&0\\ 0&0&0&\rho^{t-1}&\lambda\delta\\ 0&0&0&0&\alpha^{s-1}\end{pmatrix}} & Q_6^{(0)} & \xrightarrow{\begin{pmatrix}\lambda\delta&\rho^{t-1}&0&0&0\\ 0&\beta&-\alpha^{s-1}&0&0\\ 0&0&\lambda\delta b&\rho&0\\ 0&0&0&\beta l&-\alpha\end{pmatrix}} & Q_5^{(0)} \\
\begin{pmatrix}0&0&0&0&\delta(s,2)&0\\ 0&0&0&\delta(s,2)&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0&0&0&\delta(s,2)&0\\ 0&0&-\alpha^{s-2}&0&0\\ 0&1&0&0&0\\ -1&0&0&0&0\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0&0&0&\delta(s,2)\cdot\lambda\\ 0&0&\delta(s,2)&0\\ 0&1&0&0\\ 1&0&0&0\end{pmatrix}\Big\downarrow \\
Q_6^{(0)} & \xrightarrow[d_5^{(0)}]{} & Q_5^{(0)} & \xrightarrow[d_4^{(0)}]{} & Q_4^{(0)}
\end{array}
$$

7.3 The Ω-translates of $\widehat{x}_2$.

$$
\begin{array}{ccccccccc}
\xrightarrow{\begin{pmatrix}\beta\lambda d&0&0&0\\ \rho&-\lambda\delta b&0&0\\ 0&\alpha&\beta&0\\ 0&0&\rho^{t-1}&-\lambda\delta\end{pmatrix}} & Q_4^{(0)} & \xrightarrow{\begin{pmatrix}\delta&-\rho^{t-1}&0&0\\ 0&\beta&\alpha^{s-1}&0\\ 0&0&\lambda\delta b&-\rho\\ 0&0&0&\beta\lambda d\end{pmatrix}} & Q_3^{(0)} & \xrightarrow{\begin{pmatrix}\rho&\lambda\delta b&0&0\\ 0&\alpha&-\beta&0\\ 0&0&\rho^{t-1}&\delta\end{pmatrix}} & Q_2^{(0)} & \xrightarrow{\begin{pmatrix}\beta&-\alpha^{s-1}&0\\ 0&\lambda\delta b&\rho\end{pmatrix}} & Q_1^{(0)} \\
& \begin{pmatrix}0&0&l&0\\ 0&0&0&1\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0&0&1&0\\ 0&0&0&1\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0&l\lambda&0\\ 0&0&1\end{pmatrix}\Big\downarrow & & (0,1)\Big\downarrow \\
\xrightarrow[d_3^{(1)}]{} & Q_3^{(1)} & \xrightarrow[d_2^{(1)}]{} & Q_2^{(1)} & \xrightarrow[d_1^{(1)}]{} & Q_1^{(1)} & \xrightarrow[d_0^{(1)}]{} & Q_0^{(1)}
\end{array}
$$

$$
\begin{array}{ccc}
Q_6^{(0)} & \xrightarrow{\begin{pmatrix}\lambda\delta&\rho^{t-1}&0&0&0\\ 0&\beta&-\alpha^{s-1}&0&0\\ 0&0&\lambda\delta b&\rho&0\\ 0&0&0&\beta l&-\alpha\end{pmatrix}} & Q_5^{(0)} \\
\begin{pmatrix}0&\delta(k,1)&0&0&0\\ 0&0&l&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0&\delta(k,1)&0&0\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix}\Big\downarrow \\
Q_5^{(1)} & \xrightarrow[d_4^{(1)}]{} & Q_4^{(1)}
\end{array}
$$

7.4 The Ω-translates of $\widehat{x}_3$.

$$
\begin{array}{ccccccccc}
Q_5^{(1)} & \xrightarrow{\begin{pmatrix}\beta & -\alpha^{s-1} & 0 & 0\\ 0 & \lambda\delta & \rho & 0\\ 0 & 0 & \beta l & -\alpha\end{pmatrix}} & Q_4^{(1)} & \xrightarrow{\begin{pmatrix}\alpha & \beta l & 0\\ 0 & \rho^{t-1} & -\lambda\delta\end{pmatrix}} & Q_3^{(1)} & \xrightarrow{\begin{pmatrix}\lambda\delta & -\rho\\ 0 & \beta\lambda d\end{pmatrix}} & Q_2^{(1)} & \xrightarrow{\begin{pmatrix}-\beta\lambda d & 0\\ \rho^{t-1} & \delta\end{pmatrix}} & Q_1^{(1)} \\
{\scriptsize\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -\delta(t,2) & 0 & 0\end{pmatrix}}\Big\downarrow & & {\scriptsize\begin{pmatrix}0 & 0 & 1\\ 0 & \rho^{t-2} & 0\end{pmatrix}}\Big\downarrow & & {\scriptsize\begin{pmatrix}0 & -1\\ \lambda\delta(t,2) & 0\end{pmatrix}}\Big\downarrow & & {\scriptsize\begin{pmatrix}0 & 1\\ \rho^{t-2} & 0\end{pmatrix}}\Big\downarrow & & {\scriptsize(0,1)}\Big\downarrow \\
Q_4^{(1)} & \xrightarrow[d_3^{(1)}]{} & Q_3^{(1)} & \xrightarrow[d_2^{(1)}]{} & Q_2^{(1)} & \xrightarrow[d_1^{(1)}]{} & Q_1^{(1)} & \xrightarrow[d_0^{(1)}]{} & Q_0^{(1)}
\end{array}
$$

7.5 The Ω-translates of $\widehat{x}_4$.

$$
\begin{array}{ccccccccc}
 & \xrightarrow{\begin{pmatrix}\beta & -\alpha^{s-1} & 0 & 0\\ 0 & \lambda\delta & \rho & 0\\ 0 & 0 & \beta l & -\alpha\end{pmatrix}} & Q_4^{(1)} & \xrightarrow{\begin{pmatrix}\alpha & \beta l & 0\\ 0 & \rho^{t-1} & -\lambda\delta\end{pmatrix}} & Q_3^{(1)} & \xrightarrow{\begin{pmatrix}\lambda\delta & -\rho\\ 0 & \beta\lambda d\end{pmatrix}} & Q_2^{(1)} & \xrightarrow{\begin{pmatrix}-\beta\lambda d & 0\\ \rho^{t-1} & \delta\end{pmatrix}} & Q_1^{(1)} \\
 & & {\scriptsize(-l\lambda\delta,\,0,\,0)}\Big\downarrow & & {\scriptsize(-\alpha^{s-1},0)}\Big\downarrow & & {\scriptsize(-b\beta,0)}\Big\downarrow & & {\scriptsize(1,0)}\Big\downarrow \\
 & \xrightarrow[\rho]{} & Q_3^{(2)} & \xrightarrow[\beta]{} & Q_2^{(2)} & \xrightarrow[\alpha]{} & Q_1^{(2)} & \xrightarrow[\lambda]{} & Q_0^{(2)}
\end{array}
$$

$$
\begin{array}{ccccc}
Q_7^{(1)} & \xrightarrow{\begin{pmatrix}\delta & -\rho^{t-1} & 0 & 0 & 0 & 0\\ 0 & \beta & \alpha^{s-1} & 0 & 0 & 0\\ 0 & 0 & \lambda\delta & -\rho & 0 & 0\\ 0 & 0 & 0 & \beta l & \alpha & 0\\ 0 & 0 & 0 & 0 & \lambda\delta b & -\rho\end{pmatrix}} & Q_6^{(1)} & \xrightarrow{\begin{pmatrix}\rho & \lambda\delta b & 0 & 0 & 0\\ 0 & \alpha & -\beta l & 0 & 0\\ 0 & 0 & \rho^{t-1} & \lambda\delta & 0\\ 0 & 0 & 0 & \alpha^{s-1} & -\beta\end{pmatrix}} & Q_5^{(1)} \\
{\scriptsize(-1,0,0,0,0,0)}\Big\downarrow & & {\scriptsize(-b\beta\lambda,0,0,0,0)}\Big\downarrow & & {\scriptsize(-\rho^{t-1},0,0,0)}\Big\downarrow \\
Q_6^{(2)} & \xrightarrow[(\delta\beta\lambda)^k]{} & Q_5^{(2)} & \xrightarrow[\delta]{} & Q_4^{(2)}
\end{array}
$$

7.6 The Ω-translates of $\widehat{x}_5$.

$$
\begin{array}{ccccccccccccc}
Q_7^{(2)} & \xrightarrow{\lambda} & Q_6^{(2)} & \xrightarrow{(\delta\beta\lambda)^k} & Q_5^{(2)} & \xrightarrow{\delta} & Q_4^{(2)} & \xrightarrow{\rho} & Q_3^{(2)} & \xrightarrow{\beta} & Q_2^{(2)} & \xrightarrow{\alpha} & Q_1^{(2)} \\
{\scriptsize\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}}\Big\downarrow & & {\scriptsize\begin{pmatrix}\delta\\0\\0\\0\end{pmatrix}}\Big\downarrow & & {\scriptsize\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}\Big\downarrow & & {\scriptsize\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}\Big\downarrow & & {\scriptsize\begin{pmatrix}1\\0\\0\end{pmatrix}}\Big\downarrow & & {\scriptsize\begin{pmatrix}1\\0\end{pmatrix}}\Big\downarrow & & {\scriptsize\mathrm{id}}\Big\downarrow \\
Q_6^{(0)} & \xrightarrow[d_5^{(0)}]{} & Q_5^{(0)} & \xrightarrow[d_4^{(0)}]{} & Q_4^{(0)} & \xrightarrow[d_3^{(0)}]{} & Q_3^{(0)} & \xrightarrow[d_2^{(0)}]{} & Q_2^{(0)} & \xrightarrow[d_1^{(0)}]{} & Q_1^{(0)} & \xrightarrow[d_0^{(0)}]{} & Q_0^{(0)}
\end{array}
$$

7.7 The Ω-translates of $\widehat{y}_1$.

$$\begin{array}{ccccccc}
Q_5^{(0)} & \xrightarrow{\begin{pmatrix}\beta\lambda d & 0 & 0 & 0\\ \rho & -\lambda\delta b & 0 & 0\\ 0 & \alpha & \beta & 0\\ 0 & 0 & \rho^{t-1} & -\lambda\delta\end{pmatrix}} & Q_4^{(0)} & \xrightarrow{\begin{pmatrix}\delta & -\rho^{t-1} & 0 & 0\\ 0 & \beta & \alpha^{s-1} & 0\\ 0 & 0 & \lambda\delta b & -\rho\\ 0 & 0 & 0 & \beta\lambda d\end{pmatrix}} & Q_3^{(0)} & \xrightarrow{\begin{pmatrix}\rho & \lambda\delta b & 0 & 0\\ 0 & \alpha & -\beta & 0\\ 0 & 0 & \rho^{t-1} & \delta\end{pmatrix}} & Q_2^{(0)} \\
\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & \lambda\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\Big\downarrow & & (0,-1,0)\Big\downarrow \\
Q_3^{(0)} & \xrightarrow[d_2^{(0)}]{} & Q_2^{(0)} & \xrightarrow[d_1^{(0)}]{} & Q_1^{(0)} & \xrightarrow[d_0^{(0)}]{} & Q_0^{(0)}
\end{array}$$

7.8 The Ω-translates of $\widehat{y}_2$.

$$\begin{array}{ccccccc}
Q_5^{(1)} & \xrightarrow{\begin{pmatrix}\beta & -\alpha^{s-1} & 0 & 0\\ 0 & \lambda\delta & \rho & 0\\ 0 & 0 & \beta l & -\alpha\end{pmatrix}} & Q_4^{(1)} & \xrightarrow{\begin{pmatrix}\alpha & \beta l & 0\\ 0 & \rho^{t-1} & -\lambda\delta\end{pmatrix}} & Q_3^{(1)} & \xrightarrow{\begin{pmatrix}\lambda\delta & -\rho\\ 0 & \beta\lambda d\end{pmatrix}} & Q_2^{(1)} \\
\begin{pmatrix}0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\Big\downarrow & & \begin{pmatrix}0 & -1 & 0\\ 0 & 0 & \lambda\end{pmatrix}\Big\downarrow & & \begin{pmatrix}\lambda & 0\\ 0 & -1\end{pmatrix}\Big\downarrow & & (1,0)\Big\downarrow \\
Q_3^{(1)} & \xrightarrow[d_2^{(1)}]{} & Q_2^{(1)} & \xrightarrow[d_1^{(1)}]{} & Q_1^{(1)} & \xrightarrow[d_0^{(1)}]{} & Q_0^{(1)}
\end{array}$$

7.9 The Ω-translates of $\widehat{z}$.

$$\begin{array}{ccccccc}
 & \xrightarrow{\begin{pmatrix}\rho & \lambda\delta b & 0 & 0 & 0\\ 0 & \alpha & -\beta l & 0 & 0\\ 0 & 0 & \rho^{t-1} & \lambda\delta & 0\\ 0 & 0 & 0 & \alpha^{s-1} & -\beta\end{pmatrix}} & Q_5^{(1)} & \xrightarrow{\begin{pmatrix}\beta & -\alpha^{s-1} & 0 & 0\\ 0 & \lambda\delta & \rho & 0\\ 0 & 0 & \beta l & -\alpha\end{pmatrix}} & Q_4^{(1)} & \xrightarrow{\begin{pmatrix}\alpha & \beta l & 0\\ 0 & \rho^{t-1} & -\lambda\delta\end{pmatrix}} & Q_3^{(1)} \\
 & & \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \delta(k,1) & 0\end{pmatrix}\Big\downarrow & & \begin{pmatrix}1 & 0 & 0\\ 0 & b & 0\end{pmatrix}\Big\downarrow & & (1,0)\Big\downarrow \\
 & \xrightarrow[d_2^{(0)}]{} & Q_2^{(0)} & \xrightarrow[d_1^{(0)}]{} & Q_1^{(0)} & \xrightarrow[d_0^{(0)}]{} & Q_0^{(0)}
\end{array}$$

$$\begin{array}{ccc}
Q_7^{(1)} & \xrightarrow{\begin{pmatrix}\delta & -\rho^{t-1} & 0 & 0 & 0 & 0\\ 0 & \beta & \alpha^{s-1} & 0 & 0 & 0\\ 0 & 0 & \lambda\delta & -\rho & 0 & 0\\ 0 & 0 & 0 & \beta l & \alpha & 0\\ 0 & 0 & 0 & 0 & \lambda\delta b & -\rho\end{pmatrix}} & Q_6^{(1)} \\
\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \delta(k,1) & 0 & 0\end{pmatrix}\Big\downarrow & & \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & b & 0 & 0\\ 0 & 0 & 0 & \delta(k,1)\cdot\lambda & 0\end{pmatrix}\Big\downarrow \\
Q_4^{(0)} & \xrightarrow[d_3^{(0)}]{} & Q_3^{(0)}
\end{array}$$

7.10 The Ω-translates of $\widehat{w}$.

Since the module S_2 is Ω-periodic with period 6, the Ω-translates of $\widehat{w}$ are the identity maps:

$$\Omega^i(\widehat{w}) = \mathrm{id}\colon \Omega^{i+6}(S_2) \to \Omega^i(S_2),\ i \in \mathbb{N}.$$