Estimates for the homological dimensions of
pullback rings

N. V. Kosmatov

In this article all rings are assumed to have identity elements preserved by
ring homomorphisms, and all modules, unless otherwise stated, are left modules.
For a ring Λ let lgld Λ and lwd Λ denote the left global dimension of Λ and the
left weak dimension of Λ respectively. For a Λ-module X and a right Λ-module Y
we denote the projective dimension of X, the injective dimension of X, the flat
dimension of X, and the flat dimension of Y by pd_Λ X, id_Λ X, fd_Λ X and rfd_Λ Y
respectively.

Consider a commutative square of rings and ring homomorphisms

\[
\begin{align*}
R &\xrightarrow{i_1} R_1 \\
\downarrow i_2 & \downarrow j_1 \\
R_2 &\xrightarrow{j_2} R',
\end{align*}
\]

where R is the pullback (also called fibre product) of R_1 and R_2 over R', that
is, given r_1 \in R_1, r_2 \in R_2 with j_1(r_1) = j_2(r_2) there is a unique element
r \in R such that i_1(r) = r_1 and i_2(r) = r_2. From now on we assume that i_1 is a
surjection.

The fundamental tool used to estimate the homological dimensions of R is
the following theorem.

**Theorem 1.** (i) R-module M is injective if and only if R_1-module
Hom_R(R_1, M) and R_2-module Hom_R(R_2, M) are injective.
(ii) R-module M is projective if and only if R_1-module R_1 \otimes_R M and R_2-module
R_2 \otimes_R M are projective.
(iii) R-module M is flat if and only if R_1-module R_1 \otimes_R M and R_2-module
R_2 \otimes_R M are flat.

Establishing these assertions was a stimulus to study the category of R-
modules, which has been highly interesting for algebraists since the 1970s. In
1971 J. Milnor [2, Theorems 2.1, 2.2, 2.3] first proved Theorem 1 for projective
modules, assuming that j_2 is surjective. In 1985 A. N. Wiseman [3] showed that
this assumption could be dropped, and obtained the following upper bound on
the left global dimension of R :

\[
lgl R \leq \max_{k=1,2} \{lgl R_k\} + \max_{k=1,2} \{rfd R_k\}.
\]
He also pointed out the fact that it is impossible to estimate $\text{lgl}d R$ with only $\text{lgl}d R_k$ given, because there exists an example in which the pullback $R$ has infinite global dimension whilst those of the component rings $R_k$ are finite. All three statements of Theorem 1 were proved in 1985 by A. Facchini and P. Vámos [4, Theorem 2] under the assumption that $j_2$ is surjective. We shall see further that this assumption is superfluous.

In 1988 E. Kirkman and J. Kuzmanovich [5, Theorem 2] showed that if $j_2$ is surjective then

$$\text{lgl}d R \leq \max_{k=1,2} \{ \text{lgl}d R_k + \text{rf}d R_k \}. \quad (1)$$

In 1992 for commutative rings S. Scrivanti [6, Theorems 1, 2] sharpened this bound and obtained an upper bound on $\text{lwd} R$. Moreover, she gave examples to illustrate that, in a certain sense, her results were best possible.

In 1997 K. M. Cowley [7, Theorem 3.1] proved that if $j_2$ is surjective then

$$\text{lgl}d R \leq \max_{k=1,2} \{ \text{lgl}d R_k + \text{pd} R_k \}. \quad (2)$$

Here all the dimensions are concerned with the left-hand side of rings and modules, so this bound is “one sided”, and all the preceding bounds were “two-sided”. The comparison of (1) and (2) [7, Example 3.4] demonstrates that it may be beneficial to concentrate on a particular side of the rings.

The aim of this paper is to give a new “one sided” upper bound for $\text{lgl}d R$ (Theorem 5) and to generalize Scrivanti’s upper bound for $\text{lwd} R$ to the non-commutative case (Theorem 9). To do this, we estimate the injective and flat dimensions of an $R$-module (Propositions 4, 8). The bound (2) and its analogue for $\text{lwd} R$ are deduced as immediate consequences of our results (Corollaries 6, 10). Besides, we relax the conditions and do not require $j_2$ to be surjective.

We begin by showing how to dispense with this condition in Theorem 1.

**Proof of Theorem 1.** Set $R'' = j_2(R_2)$. Since $i_1$ is a surjection, we obtain $j_1(R_1) \subset j_2(R_2) = R''$. Hence we have another commutative square of rings and ring homomorphisms

$$
\begin{array}{ccc}
R & \xrightarrow{i_1} & R_1 \\
\downarrow & & \downarrow j_1 \\
R_2 & \xrightarrow{j_2} & R''
\end{array}
$$

with $j_2 : R_2 \rightarrow R''$ surjective. It is clear that $R$ is also the pullback of $R_1$ and $R_2$ over $R''$, so the desired result follows from [4, Theorem 2].

We need the following elementary consequence of [1, Proposition VI.2.1a].

**Lemma 2.** Let $\Lambda$ be a ring, $n$ be a positive integer, and let $0 \rightarrow M \rightarrow I \rightarrow K \rightarrow 0$ be a short exact sequence of $\Lambda$-modules where the module $I$ is injective and $\text{id}_\Lambda M \leq n$. Then $\text{id}_\Lambda K \leq n - 1$.

**Proposition 3.** Let $M$ be an $R$-module, $n$ be a positive integer, $k \in \{1, 2\}$, and let $0 \rightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \rightarrow \ldots$ be an injective resolution of $M$. 

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Let $K_t$ denote $\text{im}(f_{t+1})$, $t \geq 0$. Suppose that $\text{id}_{R_t}(\text{Ext}_R^t(R_k, M)) \leq n - l$ for $l = 0, 1, \ldots, n$. Then $\text{id}_{R_t}(\text{Hom}_R(R_k, K_t)) \leq n - l - 1$ for $t = 0, 1, \ldots, n - 1$.

**Proof.** The proof is by induction on $t$.

For $t = 0$, if we apply the functor $\text{Ext}_R^t(R_k, -)$ to the short exact sequence of $R$-modules $0 \to M \to I_0 \xrightarrow{f_1} K_0 \to 0$, we obtain an exact sequence of $R_k$-modules

$$0 \to \text{Hom}_R(R_k, M) \to \text{Hom}_R(R_k, I_0) \xrightarrow{f_1} \text{Hom}_R(R_k, K_0) \to \text{Ext}_R^1(R_k, M) \to 0$$

and isomorphisms of $R_k$-modules

$$\text{Ext}_R^1(R_k, K_0) \simeq \text{Ext}_R^{t+1}(R_k, M), \quad t \geq 1. \quad (3)$$

Setting $A_{k,0} = \text{im} f_1$, we get two short exact sequences of $R_k$-modules:

$$0 \to \text{Hom}_R(R_k, M) \to \text{Hom}_R(R_k, I_0) \xrightarrow{f_1} A_{k,0} \to 0, \quad (4)$$

$$0 \to A_{k,0} \to \text{Hom}_R(R_k, K_0) \to \text{Ext}_R^1(R_k, M) \to 0. \quad (5)$$

By Theorem 1, the $R_k$-module $\text{Hom}_R(R_k, I_0)$ is injective. Since $\text{id}_{R_k}(\text{Hom}_R(R_k, M)) \leq n$, applying Lemma 2 to (4) gives $\text{id}_{R_k}(A_{k,0}) \leq n - 1$. At the same time $\text{id}_{R_k}(\text{Ext}_R^1(R_k, M)) \leq n - 1$. Therefore, using (5), we get $\text{id}_{R_k}(\text{Hom}_R(R_k, K_0)) \leq n - 1$.

For $t \geq 1$, we apply the functor $\text{Ext}_R^t(R_k, -)$ to the short exact sequence of $R$-modules $0 \to K_{t-1} \to I_t \xrightarrow{f_{t+1}} K_t \to 0$. We obtain an exact sequence of $R_k$-modules

$$0 \to \text{Hom}_R(R_k, K_{t-1}) \to \text{Hom}_R(R_k, I_t) \xrightarrow{(f_{t+1})_*} \text{Hom}_R(R_k, K_t) \to \text{Ext}_R^1(R_k, K_{t-1}) \to 0$$

and isomorphisms of $R_k$-modules

$$\text{Ext}_R^1(R_k, K_t) \simeq \text{Ext}_R^{t+1}(R_k, K_{t-1}), \quad t \geq 1. \quad (6)$$

Put $A_{k,t} = \text{im} (f_{t+1})_*$, and consider two short exact sequences of $R_k$-modules:

$$0 \to \text{Hom}_R(R_k, K_{t-1}) \to \text{Hom}_R(R_k, I_t) \xrightarrow{(f_{t+1})_*} A_{k,t} \to 0, \quad (7)$$

$$0 \to A_{k,t} \to \text{Hom}_R(R_k, K_t) \to \text{Ext}_R^1(R_k, K_{t-1}) \to 0. \quad (8)$$

By the inductive hypothesis, we have $\text{id}_{R_k}(\text{Hom}_R(R_k, K_{t-1})) \leq n - t$. By Theorem 1, $\text{Hom}_R(R_k, I_t)$ is an injective $R_k$-module. Applying Lemma 2 to (7), we see that $\text{id}_{R_k}(A_{k,t}) \leq n - t - 1$. Combining (3) and (6) gives $\text{Ext}_R^1(R_k, K_{t-1}) \simeq \ldots \simeq \text{Ext}_R^t(R_k, K_0) \simeq \text{Ext}_R^{t+1}(R_k, M)$. Hence $\text{id}_{R_k}(\text{Ext}_R^1(R_k, K_{t-1})) = \text{id}_{R_k}(\text{Ext}_R^{t+1}(R_k, M)) \leq n - t - 1$. Finally, from (8), we obtain $\text{id}_{R_k}(\text{Hom}_R(R_k, K_t)) \leq n - t - 1$, as required.
Proposition 4. Let $M$ be an $R$-module, $n$ be a non-negative integer. Suppose that $\text{id}_{R_k}(\text{Ext}_R^l(R_k,M)) \leq n-l$ for $l = 0, 1, \ldots, n$ and $k = 1, 2$. Then $\text{id}_R M \leq n$.

Proof. For the case $n = 0$, the result follows from Theorem 1. For $n \geq 1$, consider an injective resolution of $R$-module $M$

$$0 \rightarrow M \overset{f_0}{\rightarrow} I_0 \overset{f_1}{\rightarrow} I_1 \overset{f_2}{\rightarrow} I_2 \rightarrow \ldots$$

Write $K_t = \text{im}(f_{t+1})$ for $t \geq 0$. By Proposition 3, the $R_t$-module $\text{Hom}_R(R_k, K_{t-1})$ is injective ($k = 1, 2$). Theorem 1 now shows that $K_{n-1}$ is an injective $R$-module. Therefore $\text{id}_R M \leq n$ by [1, Proposition VI.2.1a].

Proposition 4 clearly implies the following theorem.

Theorem 5. Let $n$ be a non-negative integer, and suppose that for any $R$-module $M$ we have that

$$\text{id}_{R_k}(\text{Ext}_R^l(R_k,M)) \leq n-l \text{ for } l = 0, 1, \ldots, n \text{ and } k = 1, 2.$$  

Then $\text{lgld} R \leq n$.

Corollary 6.

$$\text{lgld} R \leq \max_{k=1,2} \{\text{lgld} R_k + \text{pd}_R R_k\}.$$  

Proof. Set $n_k = \text{lgld} R_k$, $m_k = \text{pd}_R R_k$, $N_k = n_k + m_k$ ($k = 1, 2$) and $N = \max\{N_1, N_2\}$. It can be assumed that $m_k, n_k < \infty$. Let $M$ be an $R$-module and $k \in \{1, 2\}$. Since $\text{pd}_R R_k = m_k$, we have $\text{Ext}_R^l(R_k, M) = 0$ for all $l \geq m_k + 1$. At the same time, since $\text{lgld} R_k = n_k$, we get $\text{id}_{R_k}(\text{Ext}_R^l(R_k,M)) \leq n_k = N_k - m_k \leq N_k - l \leq N - l$ for all $l = 0, 1, \ldots, m_k$. Therefore, by Proposition 4, $\text{pd}_R M \leq N$. This means that $\text{lgld} R \leq N$.

Let us state the analogous results for the flat dimension of an $R$-module $M$ and the left weak dimension of $R$.

Proposition 7. Let $M$ be an $R$-module, $n$ be a positive integer, $k \in \{1, 2\}$, and let $\ldots \rightarrow F_2 \overset{f_2}{\rightarrow} F_1 \overset{f_1}{\rightarrow} F_0 \overset{f_0}{\rightarrow} M \rightarrow 0$ be a flat resolution of $M$. Let $K_t$ denote $\ker f_t$, $t \geq 0$. Suppose that $\text{fd}_{R_k}(\text{Tor}_t^R(R_k,M)) \leq n-t$ for $t = 0, 1, \ldots, n$. Then $\text{fd}_{R_k}(R_k \otimes_R K_t) \leq n-t-1$ for $t = 0, 1, \ldots, n-1$.

Proposition 8. Let $M$ be an $R$-module, $n$ be a non-negative integer. Suppose that $\text{fd}_{R_k}(\text{Tor}_t^R(R_k,M)) \leq n-t$ for $t = 0, 1, \ldots, n$ and $k = 1, 2$. Then $\text{fd}_R M \leq n$.

Theorem 9. Let $n$ be a non-negative integer, and suppose that for any finitely generated left ideal $J$ of $R$ we have that $\text{fd}_{R_k}(\text{Tor}_t^R(R_k,R/J)) \leq n-t$ for $l = 0, 1, \ldots, n$ and $k = 1, 2$. Then $\text{lwd} R \leq n$. 

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Corollary 10.
\[ \text{lwd } R \leq \max_{k=1,2} \{ \text{lwd } R_k + \text{rfd}_R R_k \}. \]

Arguing as above, the reader will easily prove Propositions 7 and 8 if he considers a flat resolution of the \( R \)-module \( M \) and applies the functor \( \text{Tor} \) instead of \( \text{Ext} \) to the resolution. Theorem 9 follows from Proposition 8 and Auslander’s theorem:

\[ \text{lwd } R = \sup \{ \text{fd}_R(R/J) \mid J \text{ is a finitely generated left ideal of } R \}. \]

For more details we refer the reader to [8], where the similar results for the projective dimension of an \( R \)-module and the left global dimension of \( R \) are proved.

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References


