Estimates for the homological dimensions of pullback rings

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In this article all rings are assumed to have identity elements preserved by ring homomorphisms, and all modules, unless otherwise stated, are left modules. For a ring Λ let $\operatorname{lgld}\Lambda$ and $\operatorname{lwd}\Lambda$ denote the left global dimension of Λ and the left weak dimension of Λ respectively. For a Λ-module X and a right Λ-module Y we denote the projective dimension of X, the injective dimension of X, the flat dimension of X, and the flat dimension of Y by $\operatorname{pd}_\Lambda X$, $\operatorname{id}_\Lambda X$, $\operatorname{fd}_\Lambda X$ and $\operatorname{rfd}_\Lambda Y$ respectively.

Consider a commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow{j_2} & R', \end{array}$$

where R is the pullback (also called fibre product) of R_1 and R_2 over R', that is, given $r_1 \in R_1$, $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there is a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$. From now on we assume that i_1 is a surjection.

The fundamental tool used to estimate the homological dimensions of R is the following theorem.

Theorem 1. (i) *R-module M is injective if and only if R_1-module $\operatorname{Hom}_R(R_1, M)$ and R_2-module $\operatorname{Hom}_R(R_2, M)$ are injective.*
(ii) *R-module M is projective if and only if R_1-module $R_1 \otimes_R M$ and R_2-module $R_2 \otimes_R M$ are projective.*
(iii) *R-module M is flat if and only if R_1-module $R_1 \otimes_R M$ and R_2-module $R_2 \otimes_R M$ are flat.*

Establishing these assertions was a stimulus to study the category of R-modules, which has been highly interesting for algebraists since the 1970s. In 1971 J. Milnor [2, Theorems 2.1, 2.2, 2.3] first proved Theorem 1 for projective modules, assuming that j_2 is surjective. In 1985 A. N. Wiseman [3] showed that this assumption could be dropped, and obtained the following upper bound on the left global dimension of R:

$$\operatorname{lgld} R \leqslant \max_{k=1,2}\{\operatorname{lgld} R_k\} + \max_{k=1,2}\{\operatorname{rfd}_R R_k\}.$$

He also pointed out the fact that it is impossible to estimate $\operatorname{lgld} R$ with only $\operatorname{lgld} R_k$ given, because there exists an example in which the pullback R has infinite global dimension whilst those of the component rings R_k are finite. All three statements of Theorem 1 were proved in 1985 by A. Facchini and P. Vámos [4, Theorem 2] under the assumption that j_2 is surjective. We shall see further that this assumption is superfluous.

In 1988 E. Kirkman and J. Kuzmanovich [5, Theorem 2] showed that if j_2 is surjective then

$$\operatorname{lgld} R \leqslant \max_{k=1,2}\{\operatorname{lgld} R_k + \operatorname{rfd}_R R_k\}. \tag{1}$$

In 1992 for commutative rings S. Scrivanti [6, Theorems 1, 2] sharpened this bound and obtained an upper bound on $\operatorname{lwd} R$. Moreover, she gave examples to illustrate that, in a certain sense, her results were best possible.

In 1997 K. M. Cowley [7, Theorem 3.1] proved that if j_2 is surjective then

$$\operatorname{lgld} R \leqslant \max_{k=1,2}\{\operatorname{lgld} R_k + \operatorname{pd}_R R_k\}. \tag{2}$$

Here all the dimensions are concerned with the left-hand side of rings and modules, so this bound is "one sided", and all the preceding bounds were "two-sided". The comparison of (1) and (2) [7, Example 3.4] demonstrates that it may be beneficial to concentrate on a particular side of the rings.

The aim of this paper is to give a new "one sided" upper bound for $\operatorname{lgld} R$ (Theorem 5) and to generalize Scrivanti's upper bound for $\operatorname{lwd} R$ to the non-commutative case (Theorem 9). To do this, we estimate the injective and flat dimensions of an R-module (Propositions 4, 8). The bound (2) and its analogue for $\operatorname{lwd} R$ are deduced as immediate consequences of our results (Corollaries 6, 10). Besides, we relax the conditions and do not require j_2 to be surjective.

We begin by showing how to dispense with this condition in Theorem 1.
Proof of Theorem 1. Set $R'' = j_2(R_2)$. Since i_1 is a surjection, we obtain $j_1(R_1) \subset j_2(R_2) = R''$. Hence we have another commutative square of rings and ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow{j_2} & R'' \end{array}$$

with $j_2 : R_2 \to R''$ surjective. It is clear that R is also the pullback of R_1 and R_2 over R'', so the desired result follows from [4, Theorem 2].

We need the following elementary consequence of [1, Proposition VI.2.1a].

Lemma 2. *Let Λ be a ring, n be a positive integer, and let $0 \longrightarrow M \longrightarrow I \longrightarrow K \longrightarrow 0$ be a short exact sequence of Λ-modules where the module I is injective and $\operatorname{id}_\Lambda M \leqslant n$. Then $\operatorname{id}_\Lambda K \leqslant n-1$.*

Proposition 3. *Let M be an R-module, n be a positive integer, $k \in \{1, 2\}$, and let $0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \longrightarrow \dots$ be an injective resolution of M.*

Let K_t *denote* $\operatorname{im}(f_{t+1})$, $t \geqslant 0$. *Suppose that* $\operatorname{id}_{R_k}(\operatorname{Ext}^l_R(R_k, M)) \leqslant n - l$ *for* $l = 0, 1, \ldots, n$. *Then* $\operatorname{id}_{R_k}(\operatorname{Hom}_R(R_k, K_t)) \leqslant n - t - 1$ *for* $t = 0, 1, \ldots, n-1$.

Proof. The proof is by induction on t.

For $t = 0$, if we apply the functor $\operatorname{Ext}^*_R(R_k, -)$ to the short exact sequence of R-modules $0 \longrightarrow M \longrightarrow I_0 \xrightarrow{f_1} K_0 \longrightarrow 0$, we obtain an exact sequence of R_k-modules

$$\begin{array}{cccccc} 0 & \longrightarrow & \operatorname{Hom}_R(R_k, M) & \longrightarrow & \operatorname{Hom}_R(R_k, I_0) & \xrightarrow{f_{1*}} \\ & \xrightarrow{f_{1*}} & \operatorname{Hom}_R(R_k, K_0) & \longrightarrow & \operatorname{Ext}^1_R(R_k, M) & \longrightarrow \quad 0 \end{array}$$

and isomorphisms of R_k-modules

$$\operatorname{Ext}^l_R(R_k, K_0) \simeq \operatorname{Ext}^{l+1}_R(R_k, M), \ l \geqslant 1. \tag{3}$$

Setting $A_{k,0} = \operatorname{im} f_{1*}$, we get two short exact sequences of R_k-modules:

$$0 \longrightarrow \operatorname{Hom}_R(R_k, M) \longrightarrow \operatorname{Hom}_R(R_k, I_0) \xrightarrow{f_{1*}} A_{k,0} \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow A_{k,0} \hookrightarrow \operatorname{Hom}_R(R_k, K_0) \longrightarrow \operatorname{Ext}^1_R(R_k, M) \longrightarrow 0. \tag{5}$$

By Theorem 1, the R_k-module $\operatorname{Hom}_R(R_k, I_0)$ is injective. Since $\operatorname{id}_{R_k}(\operatorname{Hom}_R(R_k, M)) \leqslant n$, applying Lemma 2 to (4) gives $\operatorname{id}_{R_k}(A_{k,0}) \leqslant n - 1$. At the same time $\operatorname{id}_{R_k}(\operatorname{Ext}^1_R(R_k, M)) \leqslant n - 1$. Therefore, using (5), we get $\operatorname{id}_{R_k}(\operatorname{Hom}_R(R_k, K_0)) \leqslant n - 1$.

For $t \geqslant 1$, we apply the functor $\operatorname{Ext}^*_R(R_k, -)$ to the short exact sequence of R-modules $0 \longrightarrow K_{t-1} \hookrightarrow I_t \xrightarrow{f_{t+1}} K_t \longrightarrow 0$. We obtain an exact sequence of R_k-modules

$$\begin{array}{cccccc} 0 & \longrightarrow & \operatorname{Hom}_R(R_k, K_{t-1}) & \longrightarrow & \operatorname{Hom}_R(R_k, I_t) & \xrightarrow{(f_{t+1})_*} \\ & \xrightarrow{(f_{t+1})_*} & \operatorname{Hom}_R(R_k, K_t) & \longrightarrow & \operatorname{Ext}^1_R(R_k, K_{t-1}) & \longrightarrow \quad 0 \end{array}$$

and isomorphisms of R_k-modules

$$\operatorname{Ext}^l_R(R_k, K_t) \simeq \operatorname{Ext}^{l+1}_R(R_k, K_{t-1}), \ l \geqslant 1. \tag{6}$$

Put $A_{k,t} = \operatorname{im}(f_{t+1})_*$ and consider two short exact sequences of R_k-modules:

$$0 \longrightarrow \operatorname{Hom}_R(R_k, K_{t-1}) \longrightarrow \operatorname{Hom}_R(R_k, I_t) \xrightarrow{(f_{t+1})_*} A_{k,t} \longrightarrow 0, \tag{7}$$

$$0 \longrightarrow A_{k,t} \hookrightarrow \operatorname{Hom}_R(R_k, K_t) \longrightarrow \operatorname{Ext}^1_R(R_k, K_{t-1}) \longrightarrow 0. \tag{8}$$

By the inductive hypothesis, we have $\operatorname{id}_{R_k}(\operatorname{Hom}_R(R_k, K_{t-1})) \leqslant n - t$. By Theorem 1, $\operatorname{Hom}_R(R_k, I_t)$ is an injective R_k-module. Applying Lemma 2 to (7), we see that $\operatorname{id}_{R_k}(A_{k,t}) \leqslant n - t - 1$. Combining (3) and (6) gives $\operatorname{Ext}^1_R(R_k, K_{t-1}) \simeq \ldots \simeq \operatorname{Ext}^t_R(R_k, K_0) \simeq \operatorname{Ext}^{t+1}_R(R_k, M)$. Hence $\operatorname{id}_{R_k}(\operatorname{Ext}^1_R(R_k, K_{t-1})) = \operatorname{id}_{R_k}(\operatorname{Ext}^{t+1}_R(R_k, M)) \leqslant n - t - 1$. Finally, from (8), we obtain $\operatorname{id}_{R_k}(\operatorname{Hom}_R(R_k, K_t)) \leqslant n - t - 1$, as required.

Proposition 4. *Let M be an R-module, n be a non-negative integer. Suppose that* $\mathrm{id}_{R_k}(\mathrm{Ext}^l_R(R_k, M)) \leqslant n - l$ *for* $l = 0, 1, \ldots, n$ *and* $k = 1, 2$. *Then* $\mathrm{id}_R M \leqslant n$.

Proof. For the case $n = 0$, the result follows from Theorem 1. For $n \geqslant 1$, consider an injective resolution of R-module M

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \longrightarrow \ldots .$$

Write $K_t = \mathrm{im}(f_{t+1})$ for $t \geqslant 0$. By Proposition 3, the R_k-module $\mathrm{Hom}_R(R_k, K_{n-1})$ is injective ($k = 1, 2$). Theorem 1 now shows that K_{n-1} is an injective R-module. Therefore $\mathrm{id}_R M \leqslant n$ by [1, Proposition VI.2.1a].

Proposition 4 clearly implies the following theorem.

Theorem 5. *Let n be a non-negative integer, and suppose that for any R-module M we have that*

$$\mathrm{id}_{R_k}(\mathrm{Ext}^l_R(R_k, M)) \leqslant n - l \quad \text{for} \quad l = 0, 1, \ldots, n \quad \text{and} \quad k = 1, 2 .$$

Then $\mathrm{lgld}\, R \leqslant n$.

Corollary 6.

$$\mathrm{lgld}\, R \leqslant \max_{k=1,2} \{\mathrm{lgld}\, R_k + \mathrm{pd}_R\, R_k\}.$$

Proof. Set $n_k = \mathrm{lgld}\, R_k$, $m_k = \mathrm{pd}_R\, R_k$, $N_k = n_k + m_k$ ($k = 1, 2$) and $N = \max\{N_1, N_2\}$. It can be assumed that $m_k, n_k < \infty$. Let M be an R-module and $k \in \{1, 2\}$. Since $\mathrm{pd}_R\, R_k = m_k$, we have $\mathrm{Ext}^l_R(R_k, M) = 0$ for all $l \geqslant m_k + 1$. At the same time, since $\mathrm{lgld}\, R_k = n_k$, we get $\mathrm{id}_{R_k}(\mathrm{Ext}^l_R(R_k, M)) \leqslant n_k = N_k - m_k \leqslant N_k - l \leqslant N - l$ for all $l = 0, 1, \ldots, m_k$. Therefore, by Proposition 4, $\mathrm{pd}_R\, M \leqslant N$. This means that $\mathrm{lgld}\, R \leqslant N$.

Let us state the analogous results for the flat dimension of an R-module M and the left weak dimension of R.

Proposition 7. *Let M be an R-module, n be a positive integer, $k \in \{1, 2\}$, and let* $\ldots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$ *be a flat resolution of M. Let K_t denote* $\ker f_t$, $t \geqslant 0$. *Suppose that* $\mathrm{fd}_{R_k}(\mathrm{Tor}^R_l(R_k, M)) \leqslant n - l$ *for* $l = 0, 1, \ldots, n$. *Then* $\mathrm{fd}_{R_k}(R_k \otimes_R K_t) \leqslant n - t - 1$ *for* $t = 0, 1, \ldots, n - 1$.

Proposition 8. *Let M be an R-module, n be a non-negative integer. Suppose that* $\mathrm{fd}_{R_k}(\mathrm{Tor}^R_l(R_k, M)) \leqslant n - l$ *for* $l = 0, 1, \ldots, n$ *and* $k = 1, 2$. *Then* $\mathrm{fd}_R M \leqslant n$.

Theorem 9. *Let n be a non-negative integer, and suppose that for any finitely generated left ideal J of R we have that* $\mathrm{fd}_{R_k}(\mathrm{Tor}^R_l(R_k, R/J)) \leqslant n - l$ *for* $l = 0, 1, \ldots, n$ *and* $k = 1, 2$. *Then* $\mathrm{lwd}\, R \leqslant n$.

Corollary 10.

$$\operatorname{lwd} R \leqslant \max_{k=1,2}\{\operatorname{lwd} R_k + \operatorname{rfd}_R R_k\}.$$

Arguing as above, the reader will easily prove Propositions 7 and 8 if he considers a flat resolution of the R-module M and applies the functor Tor instead of Ext to the resolution. Theorem 9 follows from Proposition 8 and Auslander's theorem:

$$\operatorname{lwd} R = \sup\{\operatorname{fd}_R(R/J) \mid J \text{ is a finitely generated left ideal of } R\}.$$

For more details we refer the reader to [8], where the similar results for the projective dimension of an R-module and the left global dimension of R are proved.

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